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Multi-Objective Genetic Algorithm to Design Manufacturing Process Line Including Feasible and Infeasible Solutions in Neighborhood

Masahiro Arakawa and Takumi Wada

Department of Civil Engineering and Systems Management, Nagoya Institute of Technology, Gokiso, Showa, Nagoya, 466-8555, Aichi, Japan

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Abstract: This paper treats multi-objective problem for manufacturing process design. A purpose of the process design is to decide combinations of work elements assigned to different work centers. Multiple work elements are ordinarily assigned to each center. Here, infeasible solutions are easily generated by precedence relationship of work elements in process design. The number of infeasible solutions generated is ordinarily larger than that of feasible solutions generated in the process. Therefore, feasible and infeasible solutions are located in any neighborhood in solution space. It is difficult to seek high quality Pareto solutions in this problem by using conventional multi-objective evolutional algorithms. We consider that the problem includes difficulty to seek high quality solutions by the following characteristics: (1) Since infeasible solutions are resemble to good feasible solutions, many infeasible solutions which have good values of objective functions are easily sought in the search process, (2) Infeasible solutions are useful to select new variable conditions generating good feasible solutions in search process. In this study, a multi-objective genetic algorithm including local search is proposed using these characteristics. Maximum value of average operation times and maximum value of dispersion of operation time in all work centers are used as objective functions to promote productivity. The optimal weighted coefficient is introduced to control the ratio of feasible solutions to all solutions selected in crossover and selection process in the algorithm. This paper shows the effectiveness of the proposed algorithm on simple model.

Keywords: Process design, process line, feasible and infeasible solution, multi-objective genetic algorithm, mix production, simulation

1. Introduction

This paper treats multi-objective problem for process design. A purpose of process design is to decide combinations of work elements assigned to all work centers. Here, the work elements denote works to produce a single product and multiple work elements are ordinarily assigned to every center. There is possibility that infeasible solutions are generated by precedence relationship of work elements in process design. In the process of process design, the number of infeasible solutions sought is ordinarily larger than that of feasible solutions. However, there are the cases that infeasible solutions are superior to feasible solutions in terms of objective functions.

This problem is treated as multi-objective problem in order to design process line for high productivity. Especially mix production appeared in a large number of process lines is focused. The following functions are introduced as multi-objective functions to evaluate the productivity of process line for mix production: the maximum value of average operation times in all work centers and the maximum value of dispersion of operation times in all work centers.

However, this problem is including feasible and infeasible solutions in any neighborhood in solution

Corresponding author: Masahiro Arakawa, Department of Civil Engineering and Systems Management, Nagoya Institute of Technology. E-mail: m.arakawa@nitech.ac.jp.

space. Therefore, it is difficult to seek high quality feasible solutions by using evolutional algorithms.

In past studies related to process design, many researchers proposed heuristic methods and rule-based procedures [1-6]. Here, one of the most conventional and popular procedures for process design is the ranked positional weight technique [6]. This is a heuristic procedure that has been used to develop many designs. However, process design is focused on a single-objective problem for line balance in these studies. In addition, infeasible solutions are neglected or infeasible solutions are modified to feasible solutions in the search process. In the case of conventional genetic algorithm, lethal chromosomes are modified to normal chromosomes to prevent from generating infeasible solutions.

We consider that a problem of process design includes the following characteristic causing difficulty to seek high quality solutions:

(1) Many infeasible solutions which have good values of objective functions are easily sought in the search process, since infeasible solutions are resemble to good feasible solutions,

(2) Infeasible solutions are useful to select new variable conditions generating good feasible solutions in search process.

When new conditions are generated from existing solutions in evolutional algorithm under the condition that feasible and infeasible solutions have same possibility to be selected, Characteristic (1) causes generation of a large number of infeasible solutions in the search process.

On the other hand, a genetic operation using Characteristic (2) is expected to promote seeking high quality feasible solutions since various patterns of chromosomes are generated. Therefore, multi-objective genetic algorithm (MOGA) including a search procedure using Characteristic (2) is proposed in this paper. Both chromosomes which generate feasible and infeasible solutions are used to select as parent chromosomes in crossover process and as chromosomes preserved for next generation in the proposed algorithm. Probability to select chromosomes generating infeasible solutions is controlled in crossover and selection for next generation in the proposed algorithm. Then, high quality feasible solutions are expected to be sought.

In this paper, the characteristics of the proposed algorithm are explained and the performance of the proposed algorithm is evaluated on simple model for process design for mix production.

2. Process Design

2.1 Characteristics of Process Design and Objective Functions

Fig.2 shows schematic diagram of assignment process for process design. Process design denotes that all work elements required to manufacture products are assigned to work centers. A precedence relationship is constructed among different work elements and this relationship is affected as constraints to assign the work elements to the work centers. Therefore, when a procedure relationship is not satisfied between different work elements assigned to works, infeasible solutions are generated. Ordinarily, since multiple work elements are assigned to each single work center, infeasible solutions which take good resultant values of objective functions are easily generated.

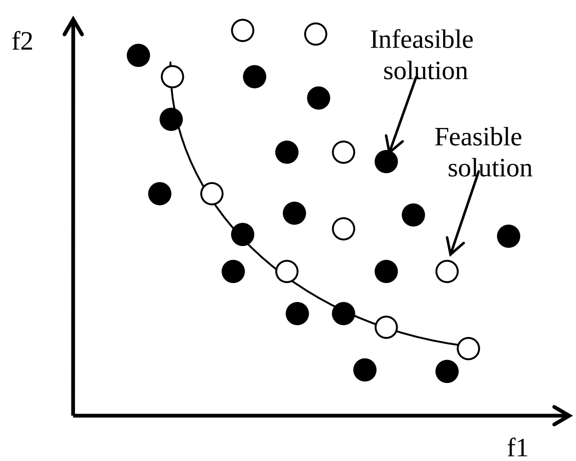

Fig. 1 Schematic diagram of search process for process design.

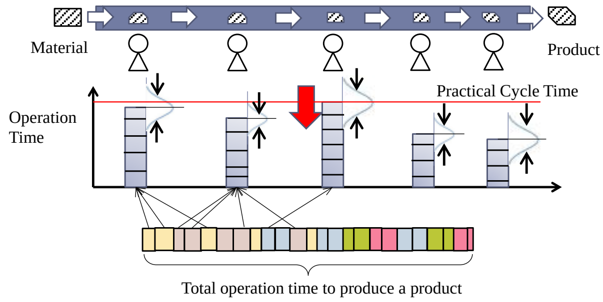

Fig. 2 Schematic diagram of assignment process for process design.

Minimum of work centers is calculated from production plan. In conventional process design, target cycle time is calculated to satisfy production plan. Here, target cycle time denotes time interval to manufacture required products from a production line in predetermined period. The cycle time is defined as the maximum operation time of work elements assigned among work centers.

The minimum number of work centers '*Nmin*' and target cycle time '*Tc*' are calculated from Eqs. (1) and (2) from production plan.

$$N_{\min} = \lceil T_{all} / T_C \rceil \tag{1}$$

$$T_C = S/Q \tag{2}$$

Here, T_{all} is total operation time to complete a product. S is planning period where products are able to be produced. Q is total amount of products required to produce. $\lceil * \rceil$ denotes floor function.

A purpose of conventional process design is to assign work elements to work centers under the condition that maximum operation time among work centers is equal to or smaller than the target cycle time. Process line calculated under the condition easily causes large difference of operation times among work centers. Since maximum operation time among work centers is practical cycle time, minimizing the practical cycle time is required to promote productivity. Large variation of operation time causes idle time at continuous work centers. Therefore, minimizing variation of operation time at all work centers is effective not to decrease productivity.

From this discussion, maximum of operation time in work centers '$Max(\mu_w)$' and dispersion of operation time in work centers '$Max(s_w)$' are introduced as objective functions to evaluate the quality of process design in this study. These functions need to minimize simultaneously to increase productivity. Eqs. (3) and (4) denote the maximum value of average operation time and the maximum value of dispersion of operation time in all work center. These are objective functions. 'm' denotes work center.

$$Max\ (\mu_m) \tag{3}$$

$$Max\ (s_m) \tag{4}$$

$$\mu_m = \sum_i \mu_i \delta_{im} \qquad \forall m \tag{5}$$

$$s_m = \sum_i s_i \delta_{im} \qquad \forall m \tag{6}$$

Here, μ_i and s_i denote average operation time and dispersion of operation time of work element 'i' respectively. δ_{im} is 0-1 variable which takes 1 when work element 'i' is assigned to work center 'm.' Otherwise, the variable takes 0.

In Fig. 2, arrows show direction of operation time at work centers to improve productivity in the process of assignment of work elements. When maximum of average operation time in all work centers is decreased by change of work elements among work centers, practical cycle time is reduced and productivity is increased. In addition, when maximum of dispersion of operation time in all work centers is decreased, standard deviation of operation time is decreased and idle time caused by variation of operation time among different work centers is reduced.

The mathematical model of this problem is written as following:

$$Min\ C_{\max} \tag{7}$$

$$Min\ D_{\max} \tag{8}$$

$$Subject\ to$$

$$C_{\max} \geq \sum_{i=1}^{I} \delta_{ik} p_i \qquad \forall k \tag{9}$$

$$D_{\max} \geq \sum_{i=1}^{I} \delta_{ik} s_i \qquad \forall k \tag{10}$$

$$\sum_{k=1}^{K} \delta_{ik} = 1 \qquad \forall i \tag{11}$$

$$k + M(\delta_{ik} - 1) \leq u_i \qquad \forall i, \forall k \tag{12}$$

$$u_i \leq K(1 - \delta_{ik}) + k \qquad \forall i, \forall k \tag{13}$$

$$u_i \leq u_j \tag{14}$$

Here, Eqs. (7) and (8) denote objective functions. C_{max} and D_{max} denote maximum value of average operation time and maximum value of dispersion of operation time among work centers respectively. *k* and *i* denote work center's number and work element's number respectively. In this model, both *k* and *i* take integer numbers from one. *K* and *I* denote the maximum number of work centers and the maximum number of work elements. In addition, u_i denotes a variable which presents work center's number to which work element *i* is assigned. *M* is a large number and it takes 2*K* in this problem. When this mathematical model is resolved, the optimal solutions are obtained and the solutions are used to compare with the proposed algorithm.

2.2 Bi-Objective Genetic Algorithm Including Evaluation of Infeasible Solutions

New type of multi-objective genetic algorithm is proposed to resolve bi-objective problem for process design in this study. In a problem of process design, it is known that many infeasible solutions are ordinarily located nearby feasible solutions in solution space from numerical experiment in past. Therefore, it is not easy to seek approximate optimal Pareto solutions by using conventional multi-objective genetic algorithm. Many infeasible solutions are ordinarily superior to feasible solutions in terms of resultant values of objective functions. The proposed algorithm uses infeasible solutions in order to effectively seek approximate optimal Pareto solutions. Both feasible and infeasible solutions are adopted as chromosomes to select for crossover process and to preserve chromosomes for next generation. Values of criteria of infeasible solutions are modified by Eq. (15) in order to control probability of selection of infeasible solutions.

$$\text{Fitness(i)} = \text{KX*value(i)} \tag{15}$$

Here, Fitness(i) and value(i) denote fitness of chromosome '*i*' used for selection and resultant value of criterion of chromosome '*i*' respectively. KX denotes weighted coefficient to modify value of criterion of infeasible solutions. KX takes 1 or a number more than 1. A simple MOGA starts after KX is determined in the proposed algorithm. By introducing the coefficient, we consider that an excess of infeasible solutions is suppressed and solutions which have smaller values of objective functions are effectively sought. The process of the proposed MOGA is described in the following.

(1) Weighted coefficients located at left side and right side, 'KL0' and 'KR0', are determined to be initial values. Here, KL0 is smaller than KR0. Parameter to control weighted coefficient number 'α' is determined to be constant number and the maximum number of iteration to change the weighted coefficient 'MAXITER' is determined to be a constant number.

(2) The number of iteration is determined to be 0. The coefficient located at left side 'KL' and the coefficient located at right side 'KR' are calculated from Eqs.(16) and (17).

$$\text{KL} = \alpha\ \text{(KR0-KL0)+KL0} \tag{16}$$

$$\text{KR} = \text{KR0} - \alpha\ \text{(KR0-KL0)} \tag{17}$$

(3) The ranked positional weight technique is executed to generated an initial solution.

(4) Initial population is generated by using the solution generate in Step (3).

(5) The coefficient located at left side 'KL' is assigned to KX. Then, a single MOGA is executed using the coefficient determined to be KX.

(6) The coefficient located at right side 'KR' is assigned to KX. Then, a single MOGA is executed using the coefficient determined to be KX. Move to Step (8).

(7) A single MOGA is executed using the

coefficient determined to be KX.

(8) Pareto solutions obtained using the coefficient located at left side 'KL' is compared with that obtained using the coefficient located at right side 'KR.'

(9) Move to Step (10) when feasible solutions obtained by using the coefficient located at left side 'KL' are superior to feasible solutions obtained by using the coefficient located at right side 'KR.' Move to Step (11) when feasible solutions obtained by using the coefficient located at right side 'KR' are superior to solutions at the other side. Otherwise, move to Step (12).

(10) Each weighted coefficient is changed by using the following equations:

$$\begin{aligned} KR0 &= KR, \\ KR &= KL, \\ KL &= KL0+\alpha(KR0-KL0). \end{aligned} \tag{18}$$

Then, the coefficient located at left side 'KL' is assigned to KX. Move to Step (12).

(11) Each weighted coefficient is changed by using the following equations:

$$\begin{aligned} KL0 &= KL, \\ KL &= KR, \\ KR &= KR0-\alpha(KR0-KL0). \end{aligned} \tag{19}$$

Then, the coefficient located at right side 'KR' is assigned to KX.

(12) The number of iteration is added one.

(13) Complete the algorithm when the number of iterations is equal to MAXITER. Otherwise, go back to Step (7).

In this study, α takes 0.3820. Here, this search process to control one dimensional variable using this constant number is similar to 'Golden search method.' Pareto and feasible solutions obtained by using different coefficients are compared with each other in Steps (8) and (9). When the number of the Pareto and feasible solutions obtained by using the weighted coefficient KX is larger than that of the Pareto and feasible solutions obtained by using the other coefficient, the coefficient KX regards as superior condition to the other coefficient.

The process of a single MOGA in Steps (5), (6), and (7) is described in the following.

(S1) Population 'POP' and maximum generation 'MaxGen' are predetermined.

(S2) Current generation takes 0.

(S3) The weighted coefficient 'KX' is predetermined.

(S4) Crossover and mutation:

(S4-1) Select parent chromosomes by roulette selection.

(S4-2) Generate new chromosomes by crossover.

(S4-3) Investigate whether new chromosomes are identical to chromosomes included in the population or not. If a new chromosome is NOT identical to any chromosome, go to Step (S5) after the chromosome is put into the population. When there is a chromosome which is identical to a new chromosome, go to Step (S4-4) in terms of the new chromosome.

(S4-4) Execute mutation process for the new chromosome and go back to Step (S4-3).

(S5) Encode the new chromosomes and calculate values of objective functions of solutions obtained from the chromosomes.

(S6) Selection of chromosomes for new generation:

(S6-1) Select Pareto solutions;

(S6-2) Parallel selection; and

(S6-3) Roulette selection.

(S7) Judge completion of algorithm.

Complete the algorithm when the current generation is equal to the maximum number of generations 'MAXGen.' Otherwise, go back to Step (S4) after one is added to the current generation.

Population is continuously used under the condition of different weighted coefficients. Therefore, solutions are continuously improved in population while weighted coefficient is changed in the proposed algorithm.

In roulette selection in Step (S6-3), values of objective functions of a solution obtained from a chromosome are modified to one dimensional

criterion by using Eqs.(20) and (21). Furthermore, inverse of the one dimensional criterion is used as probability to select chromosomes in roulette selection because the objective functions are minimized in this problem.

$$f = \sqrt{\tilde{f}_1^{\,2} + \tilde{f}_2^{\,2}} \qquad (20)$$

$$\tilde{f}_i = \frac{f_i - f_{i\,\min}}{f_{i\,\max} - f_{i\,\min}} \qquad (21)$$

Fig. 3 shows a sample of representation of a chromosome. The chromosome is constructed of an array of work element numbers and an array of work center numbers. Numbers located in the same element number in both arrays denote that work element represented in the element is assigned to work center represented in the other element. In an array of work center number, numbers are arranged in the ascending order. In order to operate elements in an array of work center's numbers for crossover and mutation, new type of array is generated by the following modification: 1 is assigned to the last element in each partial array of same work center number and 0 is assigned to other elements in the new array. Fig. 4 denotes an example of the new array. The number of '1' included in the array is same as the number of work centers. Therefore, work centers to which work elements are assigned can be controlled by operating locations of '1' in the array.

In Step (S4-1), several types of criterion are used in roulette selection: inverse of every objective function and inverse of the one dimensional criterion defined by Eqs.(20) and (21). New chromosomes equally are generated in crossover which uses different criterion in roulette selection. When bi-objective functions in the problem are f_1 and f_2 , $1/f_1$, $1/f_2$, and $1/f$ calculated by Eqs. (20) and (21) are used as criterion in the roulette selection.

In the crossover in Step (S4-2), two-point crossover is adopted to operate elements in an array of work element numbers and an array of work center numbers independently. Two elements are arbitrarily selected in the array of two chromosomes and parts of the array between the selected elements are exchanged with each other. Lethal genes are prevented by the following process: (a) in an array of work element numbers, when numbers in the original partial array are included in the exchanged partial array, elements of the numbers in the exchanged partial array are used. The other numbers in the original partial array are rearranged in the exchanged partial array according to the order of the numbers in the original partial array. (b) in an array of work center numbers, '1' is deleted at any elements in the exchanged partial array when the number of '1' in the exchanged partial array is larger than that in the original partial array. On the other hand, '1' is added at any element in the exchanged partial array when the number of '1' in the exchanged partial array is smaller than that in the original partial array.

As for mutation process in Step (S4-4), swap is performed between any two elements by iteration of random number. Here, the random number is determined between 1 and 10 by uniform random number.

Solutions which have different chromosome and same values of objective functions are stored in population. Population is continuously used and updated in all process of the proposed algorithm. Therefore, local search is used to seek good feasible solution by changing widely the coefficient.

1	3	4	5	7	6	2	8	9	12	13	10	11
0	0	1	1	1	2	2	2	3	3	3	4	4

Fig. 3 Representation of chromosomes (Upper: A series of work element numbers, Lower: A series of work center numbers).

0	0	1	1	1	2	2	2	3	3	3	4	4
0	1	0	0	1	0	0	1	0	0	1	0	1

Fig. 4 Representation of a series of work center numbers changed for crossover and mutation (Upper: An original representation, Lower: A representation changed for crossover and mutation)

3. Numerical Experiment

3.1 Numerical conditions

In order to evaluate the performance of the proposed algorithm, process design is performed on simple model by using the proposed algorithm. In this paper, process design for mix production is discussed. The model is constructed based on mix production including three types of products named as Product A, B, and C. Here, Product A is core product. Product B is a product based on core product and several parts are exchanged. Product C is a product constructed by including additional parts in the core product. Product A, B, and C are manufactured from 11, 11, and 14 work elements respectively. Each work element is determined as assembly of a part to semi-finished product. Table 1 shows operation times of work elements. 'Av' and '#D' denote average of operation time and standard deviation of operation time respectively. Arrow to going to the left denotes that data in the column is the same as data in the left column.

Fig. 5 shows procedure diagram of work elements of all products. The numbers in circles denote work element numbers in this figure. Character 'A', 'B', and 'C' denotes Product A, B, and C. When work elements are similar to each other in mix production, the elements are combined to construct a single work element. Groups of work elements surrounded by polygons are regard as single work elements in the diagram.

The amounts of products A, B, and C to produce in production plan are 100, 220, and 100 respectively. The period for production is 8400 min. (20 days×7 hours). Table 2 shows operation times in terms of groups of work elements regarded as single work elements, and these times are calculated from Eqs. (22) and (23).

When multiple work elements are assigned to work center *r*, average operation time μ_r and dispersion of operation time s_r are calculated at the work center from Eqs. (22) and (23) considering the number of products.

$$\mu_r = \sum_i \left(P_i \sum_k \mu_{ik} \delta_{rik} \right) \Big/ \sum_i P_i \qquad (22)$$

$$s_r = \sum_i \left(P_i \sum_k s_{ik} \, \delta_{rik} \right) \Big/ \sum_i P_i \qquad (23)$$

P_i denotes the required amount of Product *i*. μ_{ik} and s_{ik} denote an average operation time and dispersion of operation time of work element *k* to produce Product *i*.

Table 1 Operation time of work elements of products (No.: work element number, Av: average time, #D: standard deviation)

Product A			Product B			Product C		
No.	Time		No.	Time		No.	Time	
	Av	#D		Av	#D		Av	#D
1	5	0.3	12	8	1.1	1	←	←
2	5	0.7	13	8	1.6	2	←	←
3	9	2.1	3	←	←	3	←	←
4	9	4.4	4	←	←	4	←	←
5	4	0.4	14	7	0.7	5	←	←
6	7	1.1	6	←	←	6	←	←
7	6	2.4	15	9	1.9	7	←	←
8	4	0.1	8	←	←	8	←	←
9	6	2.2	9	←	←	9	←	←
10	5	1.2	10	←	←	10	←	←
11	10	0.2	11	←	←	11	←	←
						16	3	0.4
						17	3	1.3
						18	4	0.2

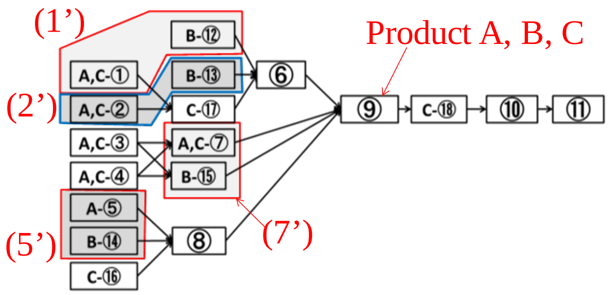

Fig. 5 Precedence diagram of work elements of all products

Table 2 Operation times of work elements combined by similarity of works (No.: work element number, Av: average time, #D: standard deviation)

No.	Av	#D	Work element number
1’	6.57	0.82	1(Product A,C), 12(Product B)
2’	6.57	1.25	2(Product A,C), 13(Product B)
5’	5.57	0.58	5(Product A,C), 14(Product B)
7’	7.57	2.15	7(Product A,C), 15(Product B)

δ_{rik} is 0—1 variable. When work element k to produce Product i is assigned to work center r, δ_{rik} takes 1. Otherwise, δ_{rik} takes 0.

The minimum number of work centers is calculated from production plan by using Eqs. (1) and (2), and the number of work centers is determined to be five. Therefore, five work centers are assumed to be located in process line in this problem. As for genetic operator used in the proposed algorithm, both population and the maximum generations are determined to be 100.

3.2 Evaluation of the Performance of the Proposed Algorithm

Production lines designed by the proposed algorithm are compared with the lines designed by the different methods. Different methods and characteristics of designed lines are shown as follows:

(D1) A production line designed by using the proposed algorithm. Here, in terms of weighted coefficients to control values of criteria of infeasible solutions, the initial value of the weighted coefficient on left side, KL0, and the initial value of the coefficient on right side, KR0, are determined as 0.0 and 20.0 respectively. The number of iteration to change weighted coefficient takes 15.

(D2) A production line designed by the ranked positional weight technique for process design.

(D3) A production line designed by resolving mathematical model for single objective functions presented in Section 2.1. The result is calculated by using free software `glpk.'

The method D2 is used to design process for production line aimed at satisfying production plan. A purpose of this method is that work elements are assigned to work centers below target cycle time. Since this method tends to generate large difference of operation times among work centers, productivity of the line is ordinarily inferior to that of the line designed by optimal methods. In the process design obtained by Method D2, the maximum value of average operation time takes 19.42 and the maximum value of dispersion of operation time takes 23.77 among work centers.

On the other hand, the different lines designed by Method D3 give 15.99 as the maximum value of average operation time among work centers and 19.36 as the maximum value of dispersion of operation time among work centers respectively.

Fig. 6 shows solutions obtained by Method D1. The distribution obtained by Method D1 is one of popular and common results obtained by five trials. The figure shows distribution of all solutions in the final population and the solutions obtained by Method D2 and Method D3. Filled circles and opened squares denote feasible solutions and infeasible solutions respectively. The solutions obtained by the proposed algorithm include the optimal solutions obtained for single objective functions by using Method D3. In the figure, several infeasible solutions are superior to feasible solutions and feasible solutions are generated at middle area between both axes. The proposed algorithm generates superior feasible solutions to the method of Method D2.

In order to evaluate productivity of the lines, event-driven simulation is executed by using data of

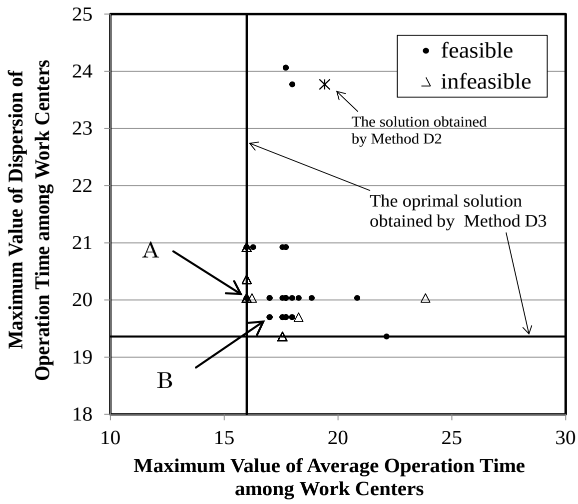

Fig.6 Distribution of solutions obtained by different methods.

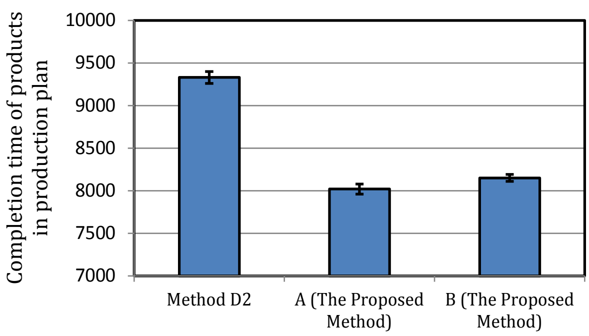

Fig.7 Comparison of completion time of products in production plannt methods

work elements assigned at each work center by different methods, because different products flow in line for mix production and difference of operation times between successive work centers tends to be large. Fig. 7 shows comparison of productivity of the lines designed by different methods. Productivity of the line is evaluated by time to complete 420 products determined in production plan. It iterates the simulation 100 times. Solution 'A' and Solution 'B' generated by the proposed algorithm are evaluated using simulation in Fig.6. Fig.7 shows that the proposed algorithm is clearly superior to Method D2 in the terms of productivity. These results show the proposed algorithm can generate Pareto and feasible solutions to provide high productivity for the problem. Since solution ‘A’ is smaller than solution ‘B’ in terms of maximum value of average operation times, average completion time of products obtained by solution ‘A’ is smaller than that obtained by the other solution. On the other hand, since solution ‘B’ is smaller than solution ‘A’ in terms of maximum value of dispersion of operation times, standard deviation of computational times obtained by solution ‘B’ is barely smaller than that obtained by the other solution.

3.3 The Characteristics of the Proposed Algorithm

In order to investigate the characteristics of the proposed algorithm in terms of the weighted coefficient, the proposed algorithm is compared with the single MOGA in the proposed algorithm under the conditions that the coefficient takes several static values between 1.0 and 50.0. Maximum generation takes 200 in order to take adequate time to seek Pareto solutions. Five trials are performed on each coefficient.

When the weighted coefficient takes 1.0, the condition denotes that both chromosomes obtaining feasible solutions and infeasible solutions have the same probability to be selected in the process of crossover and selection for next generation in the single MOGA. When the weighted coefficient takes large number such as 20.0, 30.0, and 50.0, the condition denotes that chromosomes generating feasible solutions have higher probability to be selected for crossover and selection for the next generation than chromosomes generating infeasible solutions.

Fig. 8 shows the ratio of feasible solutions to all solutions obtained by the single MOGA utilizing constant coefficients at the maximum generation. Each bar denotes average of the resultant ratios calculated from five trials and each error bar denotes standard deviation of the ratios. This figure shows that it is difficult to seek feasible solutions when the weighted coefficient takes less than 10.0. Therefore, it is difficult to seek high quality feasible and Pareto solutions under these conditions. When the weighted coefficient takes 1.0, no feasible solution is sought.

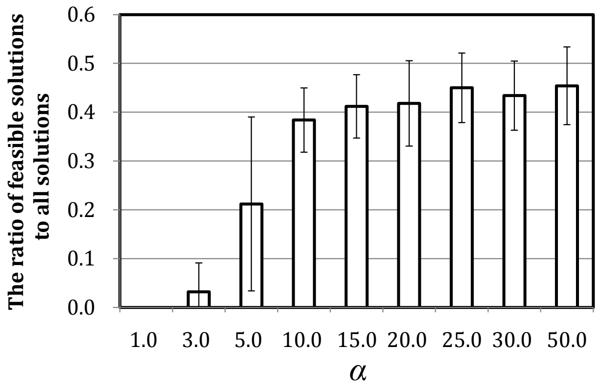

Fig. 8 The ratio of feasible solutions to all solutions obtained by the single MOGA

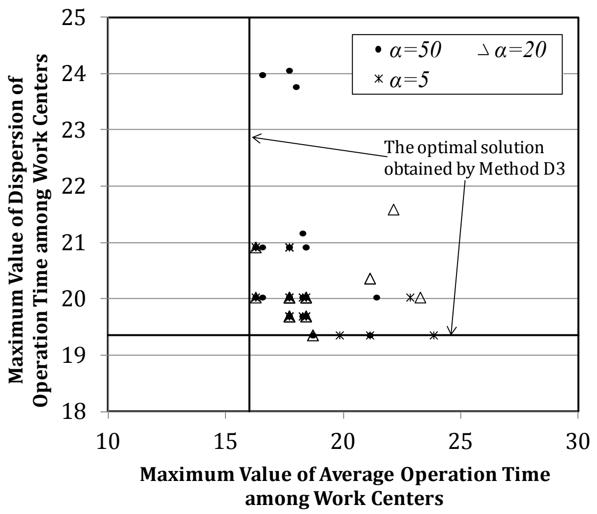

Fig. 9 Distributions of all feasible solutions obtained by the single MOGA at the maximum generation when the weighted coefficient takes different numbers.

On the other hand, when the weighted coefficient takes greater than 10.0, the ratio takes larger numbers. This result denotes that solutions are easily selected from bad feasible solutions when the coefficient takes large number. Therefore, we consider that local search to widely change the coefficient is useful to prevent from selecting solutions from bad feasible solutions.

Fig. 9 shows distributions of all feasible solutions obtained by the single MOGA at the maximum generation when the coefficient takes 5.0, 20.0, and 50.0. This figure shows the results in which the ratio of feasible solutions to all solutions takes approximately average number in five trials. The number of feasible solutions obtained from the coefficient of 50.0 is larger than that obtained from the other coefficients. In this figure, some feasible solutions are located at areas far from Pareto front when the coefficient takes 20.0 or 50.0. We consider that the resultant distribution is obtained by easily selecting feasible solutions from bad feasible solutions when the coefficient takes large numbers.

However, every coefficient generates similar solutions. This result denotes that the single MOGA is capable to seek high quality feasible and Pareto solutions when the coefficient takes large number.

The weighted coefficient which generates adequate quality feasible solutions is not clear ordinarily. Therefore, control of the coefficient included in the proposed method is effective to seek feasible and Pareto solutions for problem generating many infeasible solutions such as process design.

4. Conclusions

This study treats multi-objective problem for process design for mix production. Infeasible solutions are easily generated more than feasible solutions in the problem because precedence relationship of work elements does not easily satisfy by changing assignment of the elements slightly.

New type of a multi-objective genetic algorithm is proposed and the proposed MOGA includes local search including control of the probability to select feasible solutions while both feasible solutions and infeasible solutions are stored. The proposed MOGA is performed on simple model of process design for mix production to evaluate the algorithm. Numerical experiments show that the proposed algorithm is effective to design production line which provides high productivity as well as high quality feasible and Pareto solutions. In addition, the results show the effectiveness of control to select infeasible solutions in the proposed algorithm.

In future study, the proposed algorithm is applied to large scale and complex problem of process design. Furthermore, search process in the single MOGA is

developed as PSO and MOGA including specific local search to effectively generate high quality feasible.

Acknowledgment

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Constraints in Lean Optimization Method Tested on a Real Engineering Problem

Sven Ahlinder[1] and Olivier Goury[2]

1. Volvo ATR CTP9A 40508 Gothenburg Sweden

2. Chalmers University of Technology,SE-412 96 Gothenburg, Sweden

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Abstract: Optimizing a vehicle includes testing millions of parameters with hundreds of constraints of the performance. In this article, 162 parameters are optimized with 5 constraints using Lean Optimization combined with Linear Programming. The method converges in this example in about 100 evaluations. This is less than the gradient method needs for its first step.

Key words: Lean optimization, super saturated design, linear programming, constraints.

1. Introduction

Vehicle testing involves fabricating different vehicle parts that should be tested in different aspects. A vehicle has lots of constraints, for instance emissions, noise, space, handling and looks. This testing normally takes some months.

A vehicle is defined by millions of parameters. Volvo can not do one test vehicle per defining parameter. Due to the high number of parameters and the long response time, the Lean Optimization method has been introduced. The Lean Optimization method combined with Linear Programming of the meta-model seems to be an efficient tool in optimizing vehicles.

Lean Optimization method is a method introduced by Iana Siomina and Sven Ahlinder. It is based on the science of using experiments, Design of Experiments, DoE. In DoE, a planned and designed experimental series is performed and the steepest ascent is calculated. In DoE the number of experiments often is larger than the number of parameters. In Lean Optimization however, the number of parameters (162 in our test problem) is far more than the number of experiments for each step (14 in our case). This means that the gradient is not calculated for each parameter in the problem one at a time, but is rather to be estimated in a sub space of the problem space. This method needs much less evaluations of the problem which is important if the evaluation time is days or years and the parameters are thousands or millions.

Sven Ahlinder and Ivar Gustafsson concluded that the expected angle v between the estimated and true gradient depends on the number of rows, m, and columns, n. in the design matrix. E(cos(v))=square root(m/n) for large m, m<n. Assuming m/n=1/4 gives cos(v)=1/2 so the step taken in the approximate gradient direction gives half the gain compared to a full investigation, but allocates only 25% of the resources.

For this problem, m=162 and n=14 which gives a ratio of gain=sqrt(14/162)=29%. This means that spending just 9% of the efforts we gain 29% of the "correct" steepest ascent. The normal, gradient, estimation of the ascent is calculated by 162 experiments and will give 100% true ascent direction.

In this article constraints on both in-data and out-data are considered trying to converge to an optimal solution of a constrained problem. The linear meta-model of the problem together with the linear constraints on in and out data is fed into a Linear

Corresponding author: Sven Ahlinder, Volvo ATR CTP9A, Sweden. E-mail: Sven.Ahlinder@volvo.com.

Programming solver and a good solution is gained using only a few evaluations of the probably expensive function.

The problem is to optimize a physical object with several limitations. Since the method is not still established it could not be tested on a real physical object. The example studied here is a genuine Volvo calculation program for designing hardware for a heavy duty diesel engine. This calculation program behaves in many ways as a physical engine. That is why it is so valuable both for engine development and for this test. This engine calculation program is called "gas exchange calculation" and is very rapid (minutes) in response. Other calculation programs, Computational Fluid Dynamics for instance, have response times of weeks on a super computer.

In this calculating program, emulating a Volvo D13 engine, there are some discrete variables, but the over 100 variables considered are continuous and limited with a lower and upper value. Those are for instance bore and compression ratio of the cylinder, diameters and lengths of all tubing, timing and other injection parameters and parameters for the turbo and EGR systems. The out-data considered is brake mean efficient pressure, BMEP, the specific power of the engine, which is maximized with constraints on emissions and forces of the engine. All evaluations are made at maximum injected fuel amount for the considered engine.

The optimization of the BMEP could have been done using a normal gradient method since the response time for the program is only a few minutes. The Lean Optimization method, however, is aimed for computational fluid dynamics programs, CFD, with response times of days or weeks. There are even hand-made physical engines with response times of months or years where this can be applied. The task is to do this optimization using as few evaluations as possible, which becomes important when performing experiments where the response times are weeks or months.

2. Method

The Lean Optimization method has showed a good potential on synthetically problems [1]. It also has a good theoretical potential [2]. It has though neither been applied on a real construction problem, nor been applied to problems with constraints in the responses.

The Lean Optimization method is related to Design of Experiments, DoE, [3] since in both cases an experimental series is run on the problem. The idea of DoE is convenient since it is defining a batch of experiments that could be prepared simultaneously. This is convenient both in calculations and using physical items when all parts needed could be produced before the series of experiments begins.

In gradient method all parameters are changed one at a time in a small interval delta. Suppose the response of the changes could be written f(X), then X could be the matrix, table, where each row is one experiment and each column is one variable. X will consist of a row of zeros for the original point and under this, a diagonal of deltas. Normally each row will be evaluated a separate experiment but it is equally true to consider X as an experimental series with f(X)=Y, where each column of Y is the vector of all results, variable by variable of the experiments in X. Regarding the gradient as a system of equations XB=Y, where each row of B is the coefficients of the gradient, the ascent direction.

In Design of Experiments, DoE, X is not a diagonal matrix, but a balanced orthogonal matrix consisting of "-1" and "+1" [3]. The cause of this is to reduce noise influence in the experimental series. The DoE method gives the system "a solid kick" [3] which means that a reasonably large interval in the in-parameters is spanned by the experimental series.

DoE has become popular in numerical calculating since the system could not only be approximated by linear models, but also with interactions, x1*x2, and quadratic terms. Having a system with interaction and quadratic terms, this may lead to an optimum in fewer experimental series but the total numbers of evaluations, experiments, is using just linear terms

assumed fewer.

A Lean Optimization design could be made in the following way: Take a full factorial two level design [3] , all against all in two levels. This could be a design matrix consisting of 10 columns and 1024 rows. Transpose this table, matrix, so the observations are used as in-parameters and the parameters are used as observations. This gives a short and fat experimental design X. The first half of the columns will be identical to the last part of the columns except for the sign. Therefore [2] recommends to cut off the last half of the transposed full factorial design and then cut off a suitable number of columns more to get the right number of columns of variables. The matrix X eventually consists of 10 rows (experiments) and maximum 512 columns (variables). This design is called Ahlinder-Gustafsson-design AG-design [4] and has more or less orthogonal rows (experiments) which will be shown useful.

In this work, a matrix of 14 rows and 162 columns were used. The reason of the large number of experiments were due to a wish that every columns should not only be independent of the others but also be balanced [3], which means there are equally many “-1” as “+1” in each column. This is used for minimizing the influence of noise and may not be applicable in a calculation program.

In the Lean Optimization method, the experimental design X is a super saturated design [5] which from an evaluation point of view will give an under-determined system of equations. This system of equations is solved using minimum norm solution of Moore-Penrose inverse [2] invX=X’(XX’)-1. In Matlab, B is easily calculated as B=pinv(X)*Y.B consists of the columns of ascent directions that are not necessarily the steepest ascent but has an expected angle v to it so E(cos(v))=square root (m/n) for large m, m<n, where m is the number of rows or experiments and n is the number of columns or parameters studied [2]. This implies that if m/n=1/4, cos(v)=1/2 and the step in ascent direction taken gives half the gain compared to a full investigation, but allocates only 25% of the resources.

There are several articles written about “super saturated design” [5]. In these articles, generally the aim is to make the columns, not the rows, of the design matrix as independent as possible. This is due to that the system of equations is solved using variable selection [3] not minimum norm, as is done here.

To handle constraints of emissions and forces, a new method was introduced. It uses linear programming, LP [6]. Linear programming solves linear problems with linear constraints. If the experimental design is solved for all out-data, Brake Mean Effective Pressure, BMEP, emissions and forces, then we have a linear (not perfect) meta-model of our problem. All responses in an evaluation point, the vector e, consisting all in-parameters will form a linear function g(e)=e*B, where e*B is a set of linear functions and each element in g is a response. The first element will be the BMEP that should be minimized fulfilling constraints on the other elements in g, the emissions and forces.

BMEP is the linear target function, emissions and forces are linear constraints on out-data and the investigated area is a set of linear constraints of the in-data since the in-data has been given a solid kick. Somewhere in this linear constrained space, there is an optimum of the linear function of BMEP fulfilling the constraints and the LP solver will find it.

This reached optimum for one experimental design is followed by another step of experimental design centered around the first optimum. The optimization will continue in several steps each reaching an optimum for BMEP in the same way as the gradient search does.

3. Results and Discussion

The Lean Optimization method combined with an LP solver works well on the engine design program. This means that it will work well on equally complex problem which could be CFD calculation or even physical engines. The LP solver finds the optimal

solution of the linear problem which is the result of the Lean Optimization approach.

The result of the optimization series can be seen in Fig. 1. Each dot is a step in the Lean Optimization method. The method has converged somewhere between 100 and 150 evaluations. This means that the Lean Optimization method in this (real) case has converged before the gradient method would be able to take its first step, since it has to do 162+1 evaluations for each step for the 162 variables.

According to Fig. 1, the engine could be downsized 8% to increase BMEP 8% with constant emissions and other constraints. The piping, injection and turbo system was well optimized already thanks to hard working engineers. The industrial solution was to increase the power of the engine 8% instead of remaking the whole engine design.

The Lean Optimization method projects the problem on a linear subspace, which rank is dependant of the number of rows or experiments in the design. Both target function and constrain functions are projected on the investigated subspace. This gives linear models of both target function and constrain functions. Probably, LP solver is useful even for full rank gradient or DoE methods. In DoE, there are also non-linear models which do not give linear problems

In the relative low dimensional model with 162 in-parameters, a standard LP solver (Matlab) will work well. The LP solver will find an optimum in a corner of the convex envelop consisting of constraints of in-data and responses where BMEP has its maximum. This point is chosen as starting point of the next investigation.

It seems by the testing that using a new subspace for the new step in the ascent direction is more efficient than using the same subspace that was used in the previous step. This is made by recombine the columns in the lean design matrix. The new subspace could be randomly or analytically recombined. The difference is not measurable, possibly due to the low rank of the subspace. For the 162 variable problem only 14

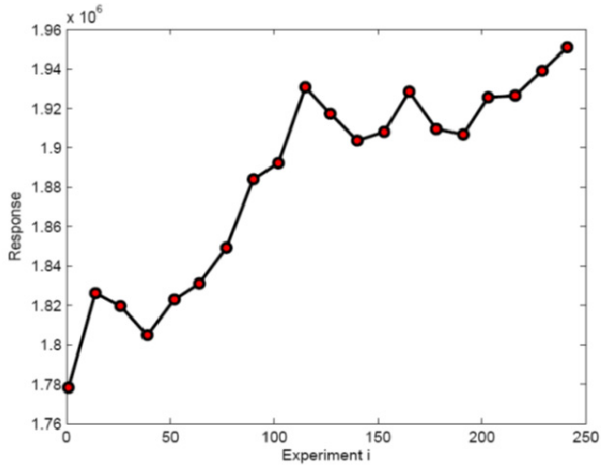

Fig. 1 Optimization result showing that Lean Optimization converges in less number of evaluations than the number of variables (162).

experiments were conducted each time. This makes the random subspace quite independent of the previous subspace. However, using the same subspace twice seems as a poor approach. This is possibly due to that the potential of this subspace is exhausted by the step taken.

When the optimum points backwards, two consecutive directions scalar product is negative, the search area was shrunken. This is generally done in DoE optimization and works well here also.

The result was that BMEP could be increased 8% just shrinking the displacement of the engine. The other over 100 variables was more or less unchanged. This is natural since those variables have been optimized by engineers in generations. The remarkable thing is that this conclusion was drawn without applying any engineering judgment, just mathematics. The result is used to increase the power 8% of the existing engine.

4. Conclusions

Lean Optimization combined with linear programming seems to be a powerful tool in constrained optimization of complex functions such as engine hardware. It is demanding to calculate each step, but when the evaluation time of the problem is days or years, the goal has to minimize the number of evaluations, not the computing time.

Physical models and physical objects could be non-linear and not well approximated with a linear model. In the engine design case, the physical based model is reasonably linear so it is useful with a linear approximation.

Shrinking the search area when having a negative consecutive directions scalar product, is a useful method also in Lean Optimization method.

It seems vital to permutate the columns of the lean design matrix to search new dimensions of the parameter space for each step.

In this example, the Lean Optimization method converges before the gradient method would have taken its first step.)

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Polynomial Quasisolutions Method for Some Linear Differential Difference Equations of Mixed Type

Valery Cherepennikov[1], Natalia Gorbatskaia[2] and Polina Sorokina[3]

1. Melentiev Energy Systems Institute SB RAS, 664033 Irkutsk, Russia

2. Irkutsk State University, Irkutsk 664001, Russia

3. Baikal National University of Economics and Law, Irkutsk 664001, Russia

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Abstract: The paper considers a scalar linear differential difference equation (LDDE) of mixed type $\dot{x}(t) = (a_0 + a_1 t)x(t) + (b_0 + b_1 t)x(t-1) + (d_0 + d_1 t)x(t+1) + \bar{f}(t), t \in R, (*)$ where $\bar{f}(t) = \sum_{n=0}^{F} \bar{f}_n t^n$. This equation is investigated with the use of the method of polynomial quasisolutions based on the representation of an unknown function in the form of polynomial $x(t) = \sum_{n=0}^{N} x_n t^n$. As a result of substitution of this function into equation $(*)$, there appears a residual $\Delta(t) = O(t^N)$, for which an exact analytical representation has been obtained. In turn, this allows one to find the unknown coefficients x_n and consequently the polynomial quasisolution $x(t)$. Several examples are considered.

Key words: Diffrential difference equations, initial value problem, polynomial quasisolutions.

1. Introduction

Investigation of any process that occurs in the modern world starts as a rule with creation of its physical model. Further this model is assigned a mathematical model which is studied by various mathematical methods. In many cases the mathematical models represent the systems of differential equations. However, the structure of these equations can be so complex that it is not possible to apply directly the methods available for studying the obtained equations. Then, we transfer from the initial system to the modified system which allows the application of available methods to study it. For example, the study of nonlinear equations employs linear equations, variable parameters of equations are replaced by constant parameters, etc. The results obtained on the basis of modified mathematical model require additional studies from the view point of their correspondence to solutions of physical model. In applications a great role is played by ordinary linear differential equations for which a great number of research methods have been developed. But, unlike rather satisfactory state of the theory of ordinary differential equations detailed research into the processes that occur in the world around makes us investigate more complex equations and take into consideration the fact that variables depend not only on the states at a current time instant but on their values at the previous and subsequent time instants as well. Besides, many problems lose sense if the relationship between variables and their values at different time instants are not considered. In these cases we use functional differential equations as mathematical models. The equations that take into account prehistory of the processes first appeared in the studies conducted by I. Bernoulli and L. Euler. At the beginning of the 1940s-1950s the interest of mathematicians in the research on such differential equations increased greatly owing to the problems of the control system theory and also because these

Corresponding authors: Valery Cherepennikov, Ph.D., senior researcher, research field: functional differential equations. E-mail: vbcher@mail.ru.

equations have very interesting properties. Nevertheless, this area intensively developed only in a very limited number of aspects and by a rather limited number of mathematicians. An impetus to the study of delay differential equations by the mathematical methods was given in the monograph by A.D.Myshkis [1], that was published in 1951. Here, it is necessary to note successive fundamental studies performed by E.Pinni [2] R.Bellman and K.L.Kuk [3], N.V. Azbelev, V.P. Maximova and L.F.Rakhmatulina [4]. Currently owing to the efforts of mathematicians from different countries the theory of differential difference equations is thoroughly developed in different areas, namely in the theory of automatic control, the theory of self-oscillating systems, in the studies related to fuel combustion in the rocket engine, in long-term forecasting in the economics, in some biophysical problems and in many other fields of science and technology, whose number is persistently growing. The abundant applications encourage rapid development of the theory of functional differential equations (FDE) and currently this theory belongs to the most developing areas in mathematical analysis. Nevertheless, even in this seemingly well studied area still there are many problems solutions to which present a great interest in terms of both science and application.

This paper focuses on the linear differential difference equations (LDDEs) of mixed type.

$$\dot{x}(t) = a(t)x(t) + b(t)x(t-1) + d(t)x(t+1) + \bar{f}(t), t \in R. \quad (1)$$

Interest in the equations that contain simultaneously delay and advanced arguments is caused by a number of problems in which these equations appear as a necessary object for study [3], [5], [6]. When coefficients of the equation are constant or have some special representations, application of Euler's method allows one to obtain analytical solutions, which are generated by roots of the characteristic quasi-polynomial. The authors have no information on the solvability conditions for LDDEs in the class of analytical functions in the case when parameters of the equation are variable. When coefficients of the equation (1) are represented by polynomials, a method of polynomial quasi-solutions (PQ-solutions) has been proposed in [7]-[10]. It implies that a formal solution of the form of polynomial $x(t) = \sum_{n=0}^{N} x_n t^n$ is introduced into the investigation. Hence the term "polynomial quasi-solution" (PQ-solution) is understood in the sense that after its substitution into the initial-value problem there appears a residual $\Delta(t) = O(t^n)$, for which an exact analytical formula has been obtained. On the other hand, this allows one to find unknown coefficients x_n and, consequently, the PQ-solution itself. The paper is devoted to finding out the conditions for the existence of PQ-solutions to the initial-value problem with the initial point for some LDDEs of mixed type, as well as to the techniques of their obtaining.

2. Problem Statement

Consider the following initial problem for scalar linear differential difference equation of mixed type.

$$\dot{x}(t) = a(t)x(t) + b(t)x(t-1) + d(t)x(t+1) + \bar{f}(t), t \in R, x(0) = x_0, \quad (2)$$

where

$$a(t) = a_0 + a_1 t, b(t) = b_0 + b_1 t, d(t) = d_0 + d_1 t, \bar{f}(t) = \sum_{n=0}^{F} \bar{f}_n t^n. \quad (3)$$

Introduce the polynomial

$$x_N(t) = \sum_{n=0}^{N} x_n t^n, t \in R. \quad (4)$$

In this case

$$\dot{x}_N(t) = \sum_{n=0}^{N} n x_n t^{n-1}, \quad (5)$$

$$x_N(t-1) = \sum_{n=0}^{N} x_n (t-1)^n = \sum_{n=0}^{N} \tilde{x}_n t^n, \quad (6)$$

$$x_N(t+1) = \sum_{n=0}^{N} x_n (t+1)^n = \sum_{n=0}^{N} \hat{x}_n t^n. \quad (7)$$

Here

$$\begin{aligned} \tilde{x}_n &= \sum_{i=0}^{N-n} (-1)^i C_{n+i}^i x_{n+i}, \\ \hat{x}_n &= \sum_{i=0}^{N-n} C_{n+i}^i x_{n+i} \end{aligned} \quad (8)$$

$$C_{n+i}^i = \frac{(n+i)!}{i!\,n!}, n = \overline{1, N-1}, \tilde{x}_N = \hat{x}_N = x_N.$$

We analyze the dimensions of polynomials obtained by substituting (3)–(7) into (2). The derivative has the dimension $N-1$, the terms $a(t)x(t)$, $b(t)x(t-1)$, $d(t)x(t+1)$ have the

dimension $N+1$, and $\bar{f}(t)$ - the dimension F. On the other hand for the last coefficient x_N in (4) to be determined by the last coefficient $\bar{f}_F$ in (3), while applying the method of undetermined coefficients, it is necessary that $N=F+1$ in (4). Define the function $f(t)$ as

$$f(t)=\sum_{n=0}^{F} f_n t^n+\Delta(t), \tag{9}$$

where $f_i=\bar{f}_i, i=\overline{0, F=N-1}$, are known coefficients (at the same time the cases where some or all $f_i=0$ are not excluded), and residual $\Delta(t)=f_N t^N+f_{N+1}t^{N+1}$, f_N and f_{N+1} are unknown coefficients. Based on the introduced notations we consider the initial problem

$$\dot{x}(t)=a(t)x(t)+b(t)x(t-1)+d(t)x(t+1)+ f(t), t\in R, x(0)=x_0. \tag{10}$$

Definition 1. *Call problem* (10) *coordinated in dimension of polynomials with respect to problem* (2).

By substituting (2)–(7) and (9) into (2), by the method of undetermined coefficients we obtain

$$nx_n=\begin{cases} a_0x_0+b_0\tilde{x}_0+d_0\hat{x}_0+f_0, n=1; \\ \sum_{i=0}^{1}(a_i x_{n-1-i}+b_i\tilde{x}_{n-1-i}+d_i\hat{x}_{n-1-i})+f_{n-1}, 2\le n\le N; \end{cases} \tag{11}$$

$$0 \;=\; \sum_{i=0}^{1}(a_i x_{N-i}+b_i\tilde{x}_{N-i}+d_i\hat{x}_{N-i})+f_N, n=N+1; \tag{12}$$

$$0 \;=\; (a_1+b_1+d_1)x_N+f_{N+1}, n=N+2. \tag{13}$$

Remark 1. Since the degree of polynomial $x(t)$ is equal to $F+1$, this makes it possible to select the degree of polynomial $\bar{f}(t)$ in (3) depending on the required degree of polynomial $x(t)$, by adding a corresponding number of zero members to $\bar{f}(t)$.

Definition 2. *If there exists a polynomial of degree* $N=F+1$

$$x_N(t)=\sum_{n=0}^{N} x_n t^n, t\in R, \tag{14}$$

that identically satisfies problem (10), then this polynomial will be called a polynomial quasi-solution (PQ-solution) to problem (2).

Thus, the objective is to identify the conditions for the existence and methods for finding unknown coefficients f_N and f_{N+1}, that make it possible to represent a solution to the initial problem (10) in the form of polynomial (14).

3. Theorem of Existence of Polynomial Quasi-Solutions

According to definition 2, the issues about the existence of polynomial quasi-solutions consist in establishing the conditions, under which we can calculate the coefficients $x_n, n=\overline{0,N}$ of polynomial quasi-solution (14) and coefficients f_N and f_{N+1} of residual $\Delta_N(t)$. To this end, let us consider relations (11)–(13). Assuming that $n=1,2,\dots,(F+2)$ and taking into account (8), we rewrite these formulas in the following form

$$(-b_0+d_0-1)x_1+(b_0+d_0)x_2+(-b_0+d_0)x_3 +\dots+((-1)^N b_0+d_0)x_N = -(a_0+b_0+d_0)x_0-f_0;$$

$$(a_0+b_0-b_1+d_0+d_1)x_1+(C_2^1(-b_0+d_0)+b_1 +d_1-2)x_2+ (C_3^2(b_0+d_0)-b_1+d_1)x_3+\cdots +((-1)^{N-1}(C_N^{N-1}b_0-b_1)+d_0 +d_1)x_n = -(a_1+b_1+d_1)x_0-f_1;$$

$$(a_1+b_1+d_1)x_1+(a_0+b_0+d_0+C_2^1(-b_1 +d_1))x_2+(C_3^1(-b_0+d_0)+ C_3^2(b_1+d_1)-3)x_3+\dots+((-1)^{N-2}(C_N^{N-2}b_0 -C_N^{N-1}b_1)+ C_N^{N-2}d_0+C_N^{N-1}d_1)x_N=-f_2;$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$(a_1+b_1+d_1)x_{N-2}+(a_0+b_0+d_0+C_{N-1}^1(-b_1 +d_1))x_{N-1}+ (C_N^1(-b_0+d_0)+C_N^2(b_1+d_1)-N)x_N=-f_F;$$

$$(a_1+b_1+d_1)x_{N-1}+(a_0+b_0+d_0+C_N^1(-b_1 +d_1))x_N=-f_N;$$

$$(a_1+b_1+d_1)x_N=-f_{N+1}. \tag{15}$$

These relations represent a system of linear algebraic equations of relatively unknown $x_n, n = \overline{0, N}$ and coefficients f_N and f_{N+1} which has the following matrix form:

$$M_N x_N^* = g_N^*, \tag{16}$$

where

$$M_N = \begin{pmatrix} -b_0 + d_0 - 1 & b_0 + d_0 & \dots & (-1)^N b_0 + d_0 & 0 & 0 \\ a_0 + b_0 - b_1 + d_0 + d_1 & C_2^1(-b_0 + d_0) + b_1 + d_1 - 2 & \dots & (-1)^{N-1}(C_N^{N-1} a_0 - b_1 + C_N^{N-1} d_0 + d_1) & 0 & 0 \\ a_1 + b_1 + d_1 & a_0 + b_0 + d_0 + C_2^1(-b_1 + d_1) & \dots & (-1)^{N-2}(C_N^{N-2} b_0 - C_N^{N-1} b_1) + C_N^{N-2} d_0 + C_N^{N-1} d_1 & 0 & 0 \\ 0 & a_1 + b_1 + d_1 & \dots & (-1)^{N-3}(C_N^{N-3} b_0 - C_N^{N-2} b_1) + C_N^{N-3} d_0 + C_N^{N-2} d_1 & 0 & 0 \\ 0 & 0 & \dots & (-1)^{N-4}(C_N^{N-4} b_0 - C_N^{N-3} b_1) + C_N^{N-4} d_0 + C_N^{N-3} d_1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -C_N^1(b_0 - d_0) + C_N^2(b_1 + 2d_1) & 0 & 0 \\ 0 & 0 & \dots & a_0 + b_0 + d_0 - C_N^1(b_1 - d_1) & 1 & 0 \\ 0 & 0 & \dots & a_1 + b_1 + d_1 & 0 & 1 \end{pmatrix}, \tag{17}$$

$$x_N^* = (x_1, x_2, \dots, x_N, f_N, f_{N+1})^T, \tag{18}$$

$$g_N^* = (-(a_0 + b_0 + d_0)x_0 - f_0, -(a_1 + b_1 + d_1)x_0 - f_1, -f_2, \dots, -f_F, 0, 0)^T. \tag{19}$$

The following theorem is true.

Theorem 1. *Let matrix* M_N *of linear system* (16) *which is determined by virtue of equation* (2)*, be nonsingular. Then for any* $x_0 \in R$ *the initial problem* (2)–(3) *will have a polynomial quasi-solution in the form of polynomial* (14) *of degree* $N = F + 1$ *with residual* $\Delta(t) = f_N t^N + f_{N+1} t^{N+1}$.

Indeed, in this case there exists the inverse matrix M_N^{-1} of matrix M_N and from (16) we obtain

$$x_N^* = M_N^{-1} g_N^*.$$

Hence, coefficients $x_1, x_2, \dots, x_N$ of polynomial quasi-solution (14) and coefficients f_N and f_{N+1} of residual $\Delta(t)$ are calculated uniquely, which proves the theorem.

4. Numerical Experiment

Consider the initial problem with an initial point for the following LDDE:

$$\dot{x}(t) = (-2 + 0.5t)x(t) + (1 + 0.5t)x(t - 1) - (2 + 0.5t)x(t + 1),$$

$$t \in R, x(0) = 1. \tag{20}$$

Taking into account (14), we determine the polynomial quasi-solution as follows

$$x_N(t) = \sum_{n=0}^{N} x_n t^n.$$

Then, according to definition 1, we write the initial problem coordinated in polynomial dimension

$$\dot{x}(t) = (-2 + 0.5t)x(t) + (1 + 0.5t)x(t - 1) - (2 + 0.5t)x(t + 1) + \Delta_N(t),$$

$$t \in R, x(0) = 1, \tag{21}$$

where $\Delta_N(t) = f_N t^N + f_{N+1} t^{N+1}$.

Consider an algorithm for finding polynomial quasi-solutions with $N = 4$. In this case linear system (16) will be written as follows:

$$M_4 x_4^* = g_4^*. \tag{22}$$

Here in accordance with (17) – (19)

$$M_4 = \begin{pmatrix} -4 & -1 & -3 & -1 & 0 & 0 \\ -4 & -8 & -4 & -12 & 0 & 0 \\ 0.5 & -0.5 & -12 & -10 & 0 & 0 \\ 0 & 0.5 & -6 & -16 & 0 & 0 \\ 0 & 0 & 0.5 & -7 & 1 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 1 \end{pmatrix},$$

$$x_4^* = (x_1, x_2, x_3, x_4, f_4, f_5)^T,$$
$$g_4^* = (3, -0.5, 0, 0, 0, 0)^T.$$

Since $\det M_4 = 5120 \neq 0$, we find the inverse matrix

$$M_4^{-1} = \begin{pmatrix} -0.2241 & -0.0170 & 0.0703 & -0.0172 & 0 & 0 \\ 0.1023 & -0.1072 & -0.0391 & 0.0984 & 0 & 0 \\ -0.0795 & 0.0680 & -0.0918 & 0.0113 & 0 & 0 \\ 0.0330 & -0.0289 & 0.0332 & -0.0637 & 0 & 0 \\ 0.2708 & -0.2360 & 0.2783 & -0.4514 & 1 & 0 \\ -0.0165 & 0.0144 & -0.0166 & 0.0318 & 0 & 1 \end{pmatrix}.$$

Then

$$x_4^* = M_4^{-1} g_4^* = (-0.6642, 0.3606, -0.2725, 0.1135, 0.9304, -0.0567)^T.$$

Hence, we obtain

$$x_4(t) = 1 - 0.6642t + 0.3606t^2 - 0.2725t^3 + 0.1135t^4,$$
$$\Delta_4(t) = 0.9304t^4 - 0.0567t^5.$$

Let us present the results of calculations of polynomial quasi-solutions for $N = 5 - 9$.

$$x_5(t) = 1 - 0.7134t + 0.3378t^2 - 0.1560t^3 + 0.1309t^4 - 0.0497t^5,$$
$$\Delta_5(t) = -0.4631t^5 + 0.0249t^6,$$
$$x_6(t) = 1 - 0.7264t + 0.4027t^2 - 0.1310t^3 + 0.0572t^4 - 0.0609t^5 + 0.0215t^6,$$
$$\Delta_6(t) = 0.2241t^6 - 0.0108t^7,$$
$$x_7(t) = 1 - 0.7077t + 0.4206t^2 - 0.1911t^3 + 0.0336t^4 - 0.0167t^5 + 0.02279t^6 - 0.0093t^7,$$
$$\Delta_7(t) = -0.1065t^7 - 0.0046t^8,$$
$$x_8(t) = 1 - 0.7000t + 0.389t^2 - 0.2175t^3 + 0.0836t^4 + 0.0030t^5 + 0.0023t^6 - 0.0128t^7 + 0.0040t^8,$$
$$\Delta_8(t) = 0.0509t^8 - 0.0020t^9,$$
$$x_9(t) = 1 - 0.7100t + 0.3650t^2 - 0.1777t^3 + 0.1153t^4 - 0.0353t^5 - 0.0130t^6 + 0.0018t^7 + 0.0061t^8 - 0.0018t^9,$$
$$\Delta_9(t) = -0.0246t^9 + 0.0009t^{10}.$$

The plots of polynomial quasi-solutions are given in Fig. 1.

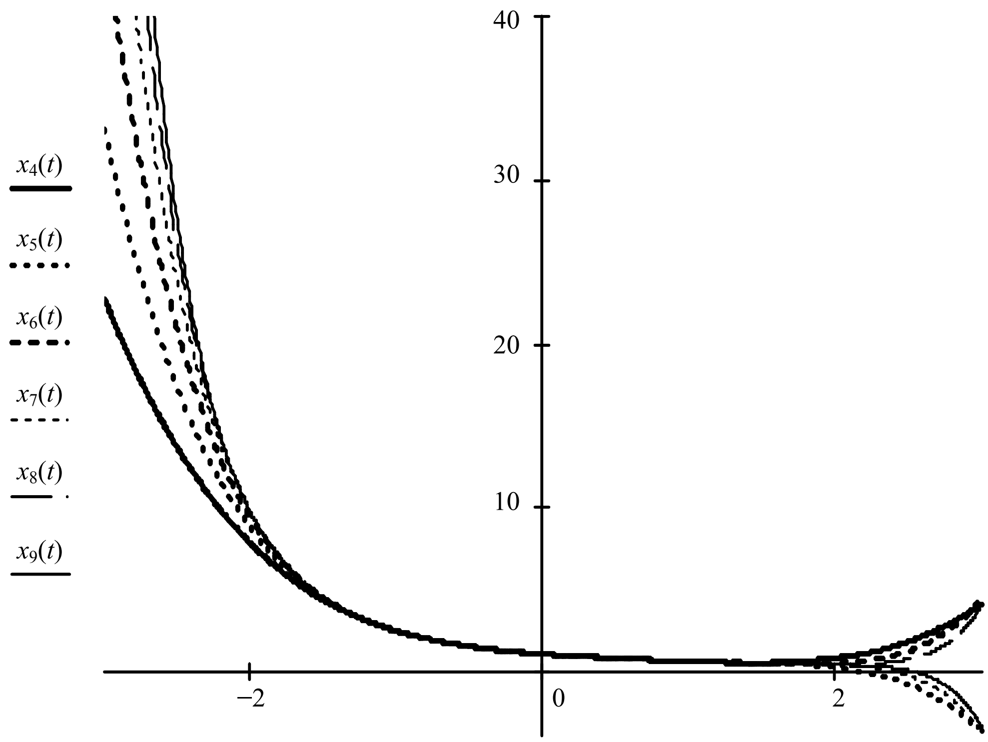

Fig. 1 Polynomial quasi-solutions

Let us introduce the following definition.

Definition 3. *ε -attractability of polynomial quasi-solutions at some interval $[t_0, t_1]$ is understood as the property of reciprocal attraction of a sequence of polynomial quasi-solutions generated by an increase in degree N of the polynomial of polynomial quasi-solutions, i.e. there exists N_* such that for $N \geq N_*$ and for a given ε*

$|x_{N+i}(t) - x_{N+i-1}(t)| < \varepsilon, i = 1,2 \ldots k, \forall t \in [t_0, t_1]$.

According to this definition, for $\varepsilon = 0.01$ polynomial quasi-solutions $x_n(t)$ of problem (20) at interval $(-1,1)$ possess the property of ε-attractability.

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People with Disabilities: Some Analyzes of the Results of the 2010 Population Census and New Challenges

Paulo Tadeu Meira e silva de Oliveira

IME-USP, Rua do Matão, 1010, Butantã, Cidade Universitária, São Paulo, Brazil

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Abstract: In his work, was applied crossings between pairs of variables, homogeneity test and technical exhaustive AID (Automatic Interaction Detection) for formation of groups second sample each of the following deficiencies: see, listen, move and intellectual from database obtained from the 2010 Population Census data sample (respondents Complete Questionnaire) formed by 20,635,472 people interviewed all over the country with the objective of studying relationship between different variables such as disability, level of education, gender, income in minimum wages among others.

Key words: Exhaustive automatic interaction detection, homogeneity test, homogeneous groups

1. Introduction

It is currently regarded as a fact that has always existed throughout history people with disabilities [1, 2]. Gradually, societies perceive that beyond the charity and assistance, such persons should be included in programs and policies that could enhance their productive potential [3, 4]. In fact, the very people with disabilities were showing signs that they could and wanted to study work and be fully included in society [5].

Thanks to the mobilize these people, it was possible to secure nowadays, a set of laws that supports populations, not only with regard to the labor market, but also as a human rights such as education, health, leisure, and finally, the right to exercise full citizenship [6].

According to the World Health Organization (WHO) estimates that more than a billion people live with some form of disability, something close to 15% of the world population (based on 2010 estimates). This is lower than the estimates coming from the IBGE, which date back to 2010 and suggest that about 23.9% of Brazil's population, are people with at least one disability.

Even with these advances, whether in historical terms, as the paradigm shift that allow people with disabilities to be citizens, as in the legal field with the existence of laws and decrees that established these rights, however, the participation of these people also is very restricted. According to the IBGE Census 2010, it is estimated that a little less than 5% had completed higher education, which indicates that a high number of people with disabilities continue exerting informal activities, poor and discontinuous or simply have no occupation, living based on retirement, pension and or family support.

It is believed that low working conditions of people with disabilities are due to situations such as difficult access to education, inadequate infrastructure, lack of information and bias on the part of schools and companies that make these people with a lower level of schooling impeding the entry of them in the formal labor market.

According to WHO [7] in regards to disabled persons notices as:

(1) Population of people with disabilities in greater

Corresponding authors: Paulo Tadeu Meira e silva de Oliveira, IME-USP, Cidade Universitária, São Paulo, Brazil. Email: poliver@usp.br.

growth driven by aging;

(2) Diverse experiences resulting from the interaction between health conditions, personal and environmental factors;

(3) Most vulnerable populations indicating higher prevalence in lower income countries;

(4) Presentation levels of poorer health;

(5) Lower educational achievement with lower rates of stay and lower school performance;

(6) Lower economic participation for people with disabilities more likely to be unemployed and earn less when employed;

(7) Higher rates of poverty, suffering high rates of deprivation with food insecurity, poor housing, lack of access to clean water and sanitation and inadequate access to health services, and finally;

(8) Greater reliance and participation restrictions due to reliance on institutional solutions, lack of community life and inadequate services.

The reverting this picture is no simple task. According to experts [5], this requires actions by at least eleven key aspects:

(1) Expanding public knowledge about people with disabilities and their integration in education and work;

(2) Issues of law (not only regarding the so-called "Quotas Law", but also the labor laws / social security);

(3) The strengthening of educational inclusion and possibilities of professional qualification and decrease the deficit amount of people with disabilities to be hired to meet the ¨ Quotas Law ¨ [6,8];

(4) More incentives to entrepreneurship triad in encouraging more free enterprise, innovation in order to encourage more creative solutions to a potential consumer market and operation of small business ideas to business, opening and closing company, niche markets in order to offer a product with a high probability of being well received and as forms capture the potential customers and have good output from potential consumers. For this, we believe that these notions points should be part of school curricula since the fundamental level so that these ideas are more mature in the minds of students in subsequent levels and complemented by the government and private enterprise with facilities for opening small business more cost subsidized financing more facilitated with lower interest rates offered by the government and financial system and longer terms of payment and existence more easy media entities linked to small and medium enterprises as SEBRAE and innovation as the USP, UNICAMP and IPEN;

(5) Accessibility as a concept summary of an inclusive society that much more than adequate infrastructure, creating conditions so that people with disabilities can use products, services and information as any other citizen;

(6) The consolidation of new paradigms and ways of thinking about disability issues in society in general, but especially between schools and employers and people with disabilities themselves;

(7) It should be considered mandatory for health professionals and care social notify the Ministry of Health and met people who have been diagnosed with problems related to their disability with CIF's;

(8) establishment of a quota of 5% of seats for admission of persons with disabilities in schools in different educational levels in order to stimulate a greater number of people with disabilities to acquire or improve their level of education;

(9) Implement methodologies for collecting data on people with disabilities, standardizes them, let them internationally comparable so that they can establish a benchmark and monitor progress and monitor policies related;

(10) strengthening the research on disability so there is increasing public understanding of the supply of information for policymaking dedicated to disability, and for the efficient allocation of resources, and ultimately, and, by end;

(11) More than Quotas Law, to ensure that the disabled person is hired for functions that correspond

to your true potential or ability [5,9].

Besides all this, it is necessary for economic and social conditions of the country evolve positively. The accelerated economic growth, better distribution of income, with quality public services and effective social programs, among other things, are beneficial to everyone, including, of course, for those with a disability. For more incredibly that it seems there isn´t a world "specific" people with disabilities.

However, to better assess the need and the fulfillment of this juridical structure, it is necessary to better describe this group of people to find answers to questions like how many? Where do they live? How do they live? What are the implications that disability involves access to all of these people human services autonomously and full? In short, as the deficiency may influence quality of life.

In terms of statistics, shows that there are few studies in formal terms, among which stands out the data obtained from the census, which allows questions as: How people with disabilities are distributed throughout the country? How to assess the access of people with disabilities in terms of the various services mentioned above? How is the evolution of disabled people by comparing them with those without disabilities? Different deficiencies are homogeneous. It is possible to form homogeneous groups. What are the variables that contribute most to the problems of deficiencies? There is a national registry of people with disabilities? Responding to these and other questions in statistical terms may possibly contribute to better support these people in order to be better assisted and resources are better managed and optimized by the actions of public policies in this area.

The Census is the most complex statistical operation undertaken by a country, when they investigated the characteristics of the population and households in the country and is the most important reference source for the knowledge of the living conditions of the population in all locations.

In Census 2010, we used two types of questionnaire: Basic Questionnaire that was administered to all households and Complete Questionnaire or Full Sample that was applied to all households selected for the sample, which besides research contained in the Basic Questionnaire, includes other characteristics of the home and search important information social, economic and demographic characteristics of its residents.

Questions relating for people with disabilities were applied in Sample Questionnaire, which sought to identify the deficiencies visual, listen and motor activity, according their degree of severity, through the perception of the people about these difficulties and for those who have declared intellectual or intellectual disability.

In terms of statistics been applied, firstly, the homogeneity test to check if the degree of severity of the types of visual impairment, of hearing and locomotion behaves similarly to the homogeneous or different levels of variables such as age, educational level and income.

Then using the technique exhaustive AID groups were formed for the study population according to each incidence of disability or not considering their degree of severity, there is at least one disability and the amount of defects considered (0-4). This method been applied to the responses of those interviewed. In each case, for each disability groups have formed for the sample population selected for Brazil at large, by region and state.

The exhaustive AID algorithm investigates all variables, all groups were scrutinized and it selects the one that presents the greatest association with the dependent variable. This is done for all independent variables and can be used in situations where the aim is to divide the population into different segments for a given criteria. In this case, the criterion chosen was the effect of various deficiencies studied.

The main objective of this work is to study the relationship between deficiencies of the population

and their ways of working and studying.

The variables used in this study were obtained directly from the Database of Sample of respondents who answered the complete questionnaire and were divided into full blocks considering the following topics: identity, family, work, instruction and disability.

2. Motivation

To be able to include people with disabilities, it is necessary, first of all, more accuracy estimate what would be the amount of people who find themselves in these conditions for each of the different disabilities, how they live and where they live, and an alternative in this case, was to consider the database obtained from census 2010 for the Sample Questionaire, and, according to this same census estimate, it is believed that there are 45,606,048 persons in Brazil with at least one permanent disability, representing approximately 23,9% of all the population represented, in the same census, by 190,755,799 habitants.

2.1 Homogeneity Test

This test consists in verifying whether a random variable behaves homogeneous in several subpopulations and fixing the sample size in each of these subsets and then selects one sample each.

To calculate the expected values, assuming that there is homogeneity between subpopulations, is used for each cell (i, j), according to expression (1) below:

$$e_{i,j} = n_i \times \frac{\text{total in the column } j}{\text{overall total}} \tag{1}$$

The total line n_i indicates the sample size of the subpopulation i, whereas the quotient, total in the column j divided by the overall total, representing the proportion of occurrences of the variable value corresponding to the column j. If there is homogeneity of the behavior of the corresponding variable, it is expected that this ratio is the same in all subpopulations.

The next step is to calculate Q^2 that is the difference between the observed and expected values using the expression (2) below:

$$Q^2 = \sum_{i=1}^{r}\sum_{j=1}^{s} \frac{o_{ij} - e_{ij}}{e_{ij}} \tag{2}$$

where r and s represent the number of row and column respectively.

To interpret the expression of Q^2, we note that the term o_{ij} - e_{ij} indicates the difference between the observed value and expected value in row i and column j, if there was homogeneity [10].

For a large number of observations, the expression of Q^2 is chi-square with (r - 1) (s - 1) degrees of freedom. The critical region contains large value of Q^2, i.e., $RC = Q^2 : Q^2 \geq q_c$, with q_c, being determined by the significance level of the test, in other words, $\alpha = P\left(Q^2 \geq q_c / H_0 \text{ is true}\right)$.

In the case of this study, homogeneity tests were made among variables that represent the different disabilities and variables such as educational level, gender and income level.

2.2. Technical exhaustive AID

It is a technique used in situations in which the goal is to divide the population into differentiated segments in relation to a given criterion. The exhaustive AID is based on chi-square test on a contingency table between the categories of the dependent variable and the categories of the independent variables [11,12]. It is a statistical method for efficient segmentation. In this particular case, the criterion chosen was the incidence of defects in each of the study [13,14].

The final objective is to build a classification tree containing only the most important variables for the classification, along with their most significant categories for the response variable. The steps of the algorithm exhaustive AID are following: i) definition of the research problem; ii) characterize the sample; iii) determining the dependent variable; iv) determination of the independent variables; v) descriptive assessment of the variables; vi) graphic AID; vii) AID

evaluation table, and finally; viii) graphical representation of the [15].

The stopping rule for the division of the groups obeys two criteria:

- when the next division result in a group with less individuals that determined (in case it was determined that the groups should have minimal, if this study, 2.5 million individuals), and;
- When there is more variable among the predictors that discriminate between the groups as determined level of significance. The choice of predictive variables is based on a test involving the chi-square statistic, and the example was set a significance level of 5%.

One advantage of this algorithm is that stop the growth of the tree before the problem of over fitting; another is that the result can be interpreted graphically. One of the disadvantages of AID exhausting algorithm is that require large amount of data to be able to ensure that the amount of observations of leaf nodes is significant [15,16]

3. Results and Discussion

This paper considers data from the Census of 2010, though, present deficiencies such as lack of other variables of interest, such as whether the disability has been acquired at birth or after a certain age who were not included in the complete questionnaire research that has been treated more comprehensively the issue of persons with disabilities. These data are useful for identifying the number of people with visual, hearing, mobility and mental, as well as its corresponding severity level to better assess and provide support for better targeting of public and private resources that can help improve the quality of life of these people.

In this study, were considered following variables allocated on the following topics:

(1) identification: ID number, state, region, household status, sex, age, race, birth registration, birth and nationality;

(2) Disability: permanent disabilities to see, hear, move and mental;

(3) Instruction: reading and writing, attends school or daycare, currently attending the course, another graduate course that attended and highest level of education;

(4) Family: the nature of marriage, marital status and number of children, and finally, and, by the end;

(5) Work: income returns home from their employment every day, time offset between home and work, occupancy condition, occupancy status, occupation position in main job and position in secondary employment.

For this study we also created the following variables with their respective categorizations: on the topic work: income in minimum wages (mw, which at the time of the completion of Census 2010 was 510 reals) categorized as 1, gain of 0 to 1 mw; 2, 1 to 3 mw; 3, 3 to 7 mw; 4, 7 to 10 mw, and, finally; 5, 10 or more mw; main job (1, registered workers; 2, civil servants and military; 3, unregistered workers; 4, self-employed; 5, employers; 6, unpaid worker, and, by the end; 7, worker in production for own consumption) the topic identification: Identification (number questionnaire or individual) and categorized age (1, if 0 to 15 years; 2 if you are aged 15 to 65, and, finally; 3, 65 or more; race(1, white; 2, black; 3, yellow; 4, mulatto, and, finally; 5, indigenous); in family topic: number of children categorized (1 for free son; 2 for the number of children between 1 and 2; 3 for many children between 3 and 5, and, finally; 4, number of children from six or more), and, finally; the topic deficiency problems: deficiencies (number of deficiencies that each individual possesses, and ranges from 0 to 4) and defic1 (0 is disabled and does not have one if you have at least one disability).

Tables 1 and 2 show the distribution of proportion in (%) for the variables total, able to read and write and instruction level in Table 1, gender, categorical age and income range categorized in Table 2 for each level of the variable disabilities.

In examining Table 1 shows the variable total 76.1%

Table 1 Distribution proportion in (%) of disabilities for the variables disabilities, total, read and write and instruction level.

disabilities	total	read and write		instruction level			
		yes	no	1	2	3	4
0	76.1	90.4	9.6	60.2	14.9	18.7	5.7
1	17.2	82.8	17.2	64.7	13.4	16.0	5.6
2	5.0	68.0	32	81.0	8.6	7.9	2.4
3	1.6	57.3	42.7	87.7	5.9	4.8	1.5
4	0.1	47.8	52.2	88.9	5.1	4.4	1.5

Table 2 Distribution in (%) of to see for the variables disabilities, total, read and write, instruction level, sex, categorical age and income class.

disabilities	sex		categorical age			income class				
	male	female	1	2	3	1	2	3	4	5
0	51.3	48.7	32.6	64.2	3.3	27.5	57.8	10.3	2.2	2.2
1	45.3	54.7	10.6	75.1	14.3	34.7	52.1	9.2	1.9	2
2	41.9	58.1	3.4	59.6	37.1	47.6	44.1	6.1	1.1	1.1
3	41.7	58.3	2.3	41.0	56.7	55.4	38.6	4.4	0.8	0.8
4	45.0	55.0	5.8	47.9	46.3	57	37.2	4	0.9	0.9

of respondents no deficiencies considered in this study; 17.2% one; 5% two; three 1.6%, and, finally; 0.1% presents four types of deficiencies studied.

Note also that the greater the number of deficiencies that the person has, the lower the proportion of those people who can read (the proportion of 90.4% of the group of people who do not have disabilities followed in decreasing rate until at 47.8% of people with disabilities the four), the greater the proportion of people attaining maximum incomplete primary level (up from 60.2% the proportion of people who do not have disabilities following at an increasing pace until it reaches 88.9% of people presenting four deficiencies) and the lower the proportion who has full fundamental level or more such as instruction level, which shows that the greater the number of shortcomings, the greater the difficulty these persons to reach a better level of education (up from 14.9% proportion with the level of education among fundamental level complete and incomplete average, 18.7% between mid-level full and incomplete higher, and finally, 5.7% with college degrees or more for people who do not have disabilities, followed in descending order until; ratio of 5.1% between fundamental level complete and incomplete secondary level, middle level between 4.4% complete and incomplete higher, and finally; proportion 1.5% with college degrees or more for people with the four disabilities). It also shows the great disparity between people who does not have disabilities and those with at least one of the deficiencies at all levels of instruction.

The results in Table 2 show that people having at least one type of deficiency is most females constituting 54.7% of individuals having deficiencies one study, 58.1% two, three 58.3%, and 55.0% four.

Regard the different age group shows that groups of people who have no more than two deficiencies are more concentrated in the age group between 15 and 65 years with 64.2% of the group who has no disability, 75.1% people who have a disability, and 59.6% of people who have two shortcomings: the group of people who have three deficiencies are more concentrated in the age above 65 years with 56.7%, and the group of people the four studied with disabilities are more concentrated in the age group between 15 to 65 years with 47.9% followed by the age group over 65 years with a proportion of 46.3%, it is also possible that the measure increases the number disabilities greater is the tendency of these people to become more concentrated in the range between 0 and 1 with a minimum wage rate of 27.5% in the group of people that has no disability and goes increasing pace until it reaches 57% of people with disabilities the four considered in the study and decreases in other income brackets, and also shows the great disparity between people who has no disability and among those with at least one of the deficiencies in all income brackets.

Tables 3 and 4 show the distributions of ratios in (%) for variables read and write and instruction level in Table 3, gender, age and income groups categorized in Table 4 for each level of the variable to see.

From the results of Table 3 show that 0.2% of respondents can´t see any way 3.3% can see with great difficulty, 15.1% can see with some difficulty, and finally 81.4% do not exhibit any deficiency in visual terms.

Table 3 Distribution in (%) of to see for the variables see, total, read and write and instruction level.

to see	total	read and write		instruction level			
		yes	no	1	2	3	4
0	0.2	69.8	31.2	71.8	10.6	12.2	4.9
1	3.3	70.3	29.7	77.2	9.9	9.9	2.8
2	15.1	81.4	18.6	68.5	12.7	15.1	5.4
3	81.4	89.2	10.8	62.5	14.2	17.5	5.5

Table 4 Distribution in (%) of to see for the variables see, sex, categorical age and income class.

to see	sex		categorical age			income class				
	male	female	1	2	3	1	2	3	4	5
0	47.7	52.3	13.6	55.8	30.6	25.2	56.4	12.1	2.9	3.4
1	45.3	54.7	5.5	63.9	30.6	45.9	46.0	5.9	1.1	1.0
2	41.9	58.1	8.0	73.4	18.6	36.0	51.1	9.1	1.9	2.0
3	41.7	58.3	31.3	64.1	4.7	28.1	57.4	10.2	2.1	2.2

It is also observed that the higher the degree of severity of the disability, the lower the proportion of people who can read and write following 89.2% of people who are not visually impaired followed in decreasing pace to reach 69.8% in people who can´t see at all.

With regard to the level of instruction is noted that the greater the severity of visual impairment less is the possibility of obtaining a better educated, except that the group of people who can´t see any way tend to have a better educated than the group of people who can see with difficulty, since the group of people who can not see any way has a higher proportion of people with education level at most elementary education and a lower proportion of people with complete primary level or more.

Table 4 generally shows a predominance of females in all different severities of visual impairment. Regarding the age groups shows a prevalence of between 15 and 65 years, with people aged over 65 years present in groups that can´t see any way and that presents a lot of difficulty with the proportion of 30.6% in both cases.

Table 4 also shows that the group of people who can´t see any way they focus more on income range between one and three minimum wages in proportion of 56.4%, the group of people who see with difficulty are more concentrated in the income range between one and three with the proportion of 46.0% followed by income range between zero and with a minimum wage proportion of 45.9%, and finally, as the group of people who see with some difficulty are more concentrated in the income range between one and three minimum wages in proportion of 51.1%.

The Tables 5 and 6 present proportions in (%) variables total, read and write and instruction level in Table 5, and; gender, age and income group categorized in Table 6 by level to listen.

From the results of Table 5 we can see that 0.2% of respondents can not hear at all, 1.0% can hear with great difficulty, 4.1% can hear with some difficulty and 94.8% did not present difficulties in terms of hearing. Note also that the more severe the disability hearing decreases the proportion of people who can read and write (the proportion of 88.2% of people who do not have hearing disabilities following in decreasing rate until reaching 62.9% group of people who can not hear at all), on the other hand, the set of people who can hear with difficulty find greater difficulty in obtaining a level of instruction rather than the group of people who fail in any way, by contain higher proportions of people with education level at most incomplete primary and lower proportions for other levels of education.

Table 5 Distribution in (%) of to listen for the variables total, read and write and instruction level.

to listen	total	read and write		instruction level				
		yes	no	1	2	3	4	5
0	0.2	62.9	37.1	76.3	9.6	10.2	3.5	0.5
1	1.0	63.8	36.2	82.6	7.8	7.4	2.2	0.1
2	4.1	73.1	26.9	75.6	10.1	10.7	3.5	0.2
3	94.8	88.2	11.8	61.7	14.4	17.9	5.6	0.5

Table 6 Distribution in (%) of to listen for the variables sex, categorical age and income class.

to listen	sex		categorical age			income class				
	male	female	1	2	3	1	2	3	4	5
0	51.0	49.0	16.4	65.4	18.2	31.2	55.2	9.1	2.2	2.3
1	53.7	46.3	5.5	45.8	48.7	44.8	46.4	6.5	1.1	1.2
2	51.0	49.0	7.0	56.4	36.6	40.2	48.6	8.0	1.6	1.7
3	49.5	50.5	27.9	66.1	6.0	29.6	56.4	10.0	2.1	2.1

In Table 6 shows that most people with hearing loss are males. In terms of age, groups of people who can´t hear at all and who do not have hearing problems are more concentrated in the range between 15 and 65 years, with the proportion of 65.4% and 66.1% respectively while groups of people who have little and very difficult to hear are concentrated in the interval between 15 and 65 years, with the proportion of 45.8% and 56.8% respectively, and older than 65 years with the proportion of 48.7% and 36.6% respectively.

Table 6 also shows that the group of people who can not hear at all can achieve better income levels than the group of people who can hear with great Difficulty, for this first group gets larger proportion of people with income class more than minimum wage and the last is more concentrated in the income class between 0 and 1 minimum wage with proportion of 44.8%.

Tables 7 and 8 show proportion in (%) of variables total, read and write and instruction level in Table 7, gender, age and categorized in income (income bracket) in table 8 per level to move.

As for getting in Table 7 shows that 0.4% fail in any way, 2% can only with great difficulty, 4.7% can with some difficulty and 92.9% has no mobility problems. In all different degrees of severity score was greater predominance of educated incomplete at most fundamental and that the proportion of people who can read and write decreases as it increases the severity of to move disability (based on the proportion of 88 9% of the group of people that has no trouble walking, followed in descending order until you reach 55% of people who can´t move at all). Note also that the group of people who can hear with difficulty is more difficult to obtain a better level of education than the group of people who can´t move at all, because the former has smaller proportions of people with level education teaching complete foundation or higher.

Table 8 also shows that the problems are getting larger in females aged over 15 years, while the group that presents no problem for to move are males aged between 15 and 65 years.

In Table 8, we can see that the groups of people who can´t move in any way and do not present any difficulty are more concentrated in the income level between one and three minimum wages, while the groups of people who able to move with little difficulty and a lot are more concentrated in the income below the minimum wage.

The Tables 9 and 10 present distribution of proportion in (%) of variables total, read and write and instruction level in Table 9, and; gender, age categorized and income group in Table 10 by level of mental.

Table 7 Distribution in (%) of to move for the variables total, read and write and instruction level.

to move	total	read and write		instruction level				
		yes	no	1	2	3	4	5
0	0.4	55.0	45.0	84.1	8.3	7.0	2.5	0.2
1	2.0	62.8	37.2	84.6	7.3	6.2	1.8	0.1
2	4.7	70.0	30.0	79.3	9.1	8.7	2.8	0.1
3	92.9	88.9	11.1	61.0	14.6	18.2	5.7	0.5

Table 8 Distribution in (%) of to move for the variables sex, categorical age and income class.

to move	sex		categorical age			income class				
	male	female	1	2	3	1	2	3	4	5
0	47.2	52.8	17.5	38.2	44.2	26,4	55.4	12	2.8	3.4
1	38.6	61.4	2.6	49.5	48.0	51	42.6	4.9	0.8	0.8
2	38.4	61.6	3.2	58.5	38.3	47,3	44.4	6	1.1	1.1
3	50.5	49.5	28.6	66.3	5.1	29	56.6	10	2.1	2.2

Table 9 Distribution in (%) of mental for the variables total, read and write and instruction level.

mental	total	read and write		instruction level				
		yes	no	1	2	3	4	5
0	1.4	48.7	51.3	86.6	6.6	5.4	1.3	0.2
1	98.6	87.8	12.2	62.1	14.3	17.6	5.5	0.4

Table 10 Distribution in (%) of mental for the variables sex, categorical age and income class.

mental	sex		categorical age			income cless				
	male	female	1	2	3	1	2	3	4	5
0	54.6	45.4	16.6	68.3	15.0	54.8	39.2	4.4	0.8	0.8
1	49.6	50.4	27.0	65.4	7.6	29.1	56.1	9.9	2.1	2.1

From Table 9 it can be noted that 1.4% of respondents have an intellectual disability with the proportion of 51.3% can´t read and write and has educated up with incomplete basic ratio of 86.6%.

In Table 10, shows that the majority of people with intellectual problems are male, with the proportion of 54.6% and aged between 15 and 65 years, with the proportion of 68.3%.

In addition, in Table 10, it appears that the majority of people with mental disabilities are more concentrated in the income level below one minimum wage ratio of 54.8%.

Making a comparison between the results of a study of the distribution of the proportions by deficiencies in levels see (Tables 3 and 4), listen (Tables 5 and 6) around (Tables 7 and 8) and mental (Tables 9 and 10) it was possible verify that:

The deficiency present in most people surveyed is the visual 18.6%, whereas this is less is the mental with only 1.4% of the people surveyed.

How to read and write a simple note the group with the worst result was formed by intellectual disability, whereas the group that presented the best result was the group formed by the visually impaired and the result was around 70% to people who can not see at all.

Note that the groups of people who can´t see, listen or move in any way get better educated and income than the group of people who can see, listen or move with very difficult, experts believe that this type of result may be due to actions of the government since the days of empire invested more in equipment and infrastructure for teaching Braille than equipment for low vision as telelupa, electronic magnifier, more wheel chair that physiotherapy treatment for can acquire more mobility and listening aid than treatment with speech therapy.

In order to evaluate the different degrees of severity yeah, can not in any way but with much difficulty, homogeneity tests were performed for these severity levels of the variables see, move and hear that in all cases were obtained level significance of 0.000 for this test and concluded that there is no homogeneity between these levels tested.

With this results checking that people who can´t in any way has better performance in terms of level of education and income than people who answered yes, but with much difficulty and experts consulted this type of situation occurs due to public actions as establishing special schools, production and import of equipment as regulate, typewriters and braille printers, these products, which benefit else who can´t in any way that anyone can, but with great difficulty, as in the case of the latter, often need special equipment and appliances just existing in the Brazilian market as telelupa low vision in the case of vision, devices that aid in hearing in case of trouble hearing and wheelchairs for people who have difficulty walking reflecting a serious economic problem for families and for the country itself, since these equipment are imported and besides the high price also adds import fees which are also high, and in most cases, are not manufactured in Brazil, and notices the nonexistence of public actions by various governments that best makes possible this kind of production and procurement.

The Tables 11a and 11b show distribution in (%) of proportions of the formal labor represented by the sum among proportions of the registered workers, civil servants and military and employers, and, informal work represented by the sum of proportions among unregistered workers, self-employed, unpaid worker and worker in production for own consumption.

From Tables 11a and 11b are possible to verify the predominance of informal work on the formal in all situations, being more pronounced in groups with intellectual disability (71.3%) who see (67.1%), listening (67.4 %) or moving (71.6%) with difficulty or having two (70.0%), three (76.0%) or four (74.4%) disabilities.

The graphs of Figs. 1 to 5 show the diagrams for AID number of deficiencies (deficiencies) and

Table 11a. Distribution proportion in (%) of the formal labor and informal work for disabilities and to see

disabilities	formal labor	informal work	to see	formal labor	informal work
0	49.7	50.3	0	51.1	48.9
1	41.5	58.5	1	32.9	67.1
2	30.0	70.0	2	41.1	58.9
3	24.0	76.0	3	49.2	50.8
4	25.6	74.4			

Table 11b. Distribution proportion in (%) of the formal labor and informal work for to listen, to move and mental.

to listen	formal labor	informal work	to move	formal labor	informal work	mental	formal labor	informal work
0	48.9	51.1	0	47.3	52.7	0	28.7	71.3
1	32.6	67.4	1	28.4	71.6	1	47.3	52.7
2	36.0	64.0	2	30.6	69.4			
3	47.6	52.4	3	48.1	51.9			

deficiencies to see, move, mental and hearing respectively. For each of these diagrams were formed six groups: the first group (black), second (red), the third (blue), the fourth (green), the fifth (brown) and sixth (pink).

Studying graphic diagram of Fig. 1 AID shows that the variables that most contribute to the appearance of a smaller number of disabilities are unmarried, economically active, attending a course and do not have secondary employment.

Among the different profiles of the groups formed what best contributes to fewer disabilities is the group formed by four unmarried and attending a course, while the profile that contributes most to the increase in the number of disabilities is the group 3 formed by being married, divorced, widowed, legally separated condition and not economically active.

For the case of Fig. 2 shows that the variables that most contribute to a lower incidence of visual impairment are unmarried, activity status, economically active status, level of education from primary level onwards and attended or attends daycare.

Considering these observations, the better is the group formed by 6 which presents unmarried, have attended daycare, education level completed elementary onwards followed by group 4 unmarried and attends

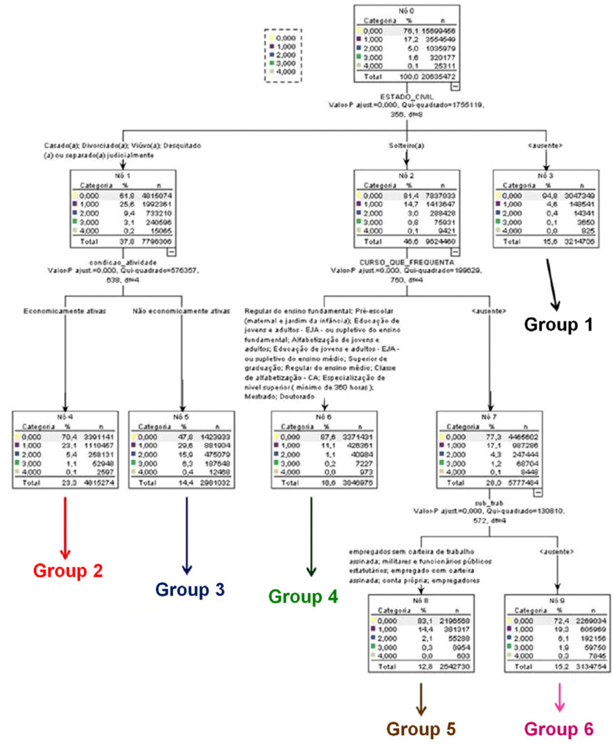

Fig. 1. Diagrama AID for dependent variable disabilities.

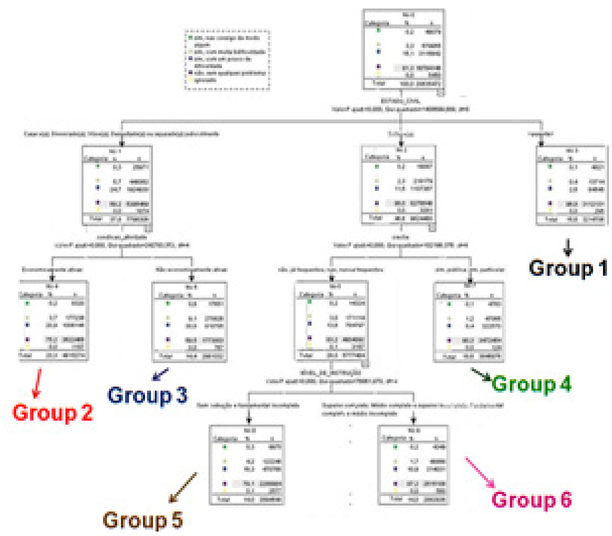

Fig. 2 Diagram AID dependent variable to see.

public or private kindergarten, while the largest contributor to the increase visual impairments is group 5 consists of being married, divorced, widowed, separated or legally separated and not economically active.

As for the Fig. 3, the variables that most contribute to a lower incidence of deficiencies for around are unmarried, economically active condition,

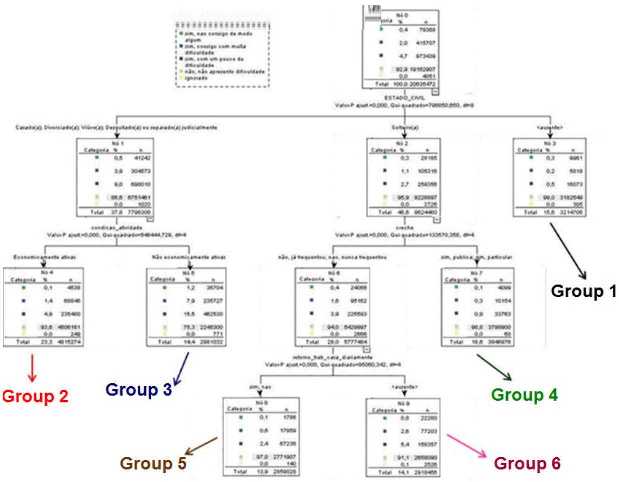

Fig. 3 Diagram AID dependent variable to move.

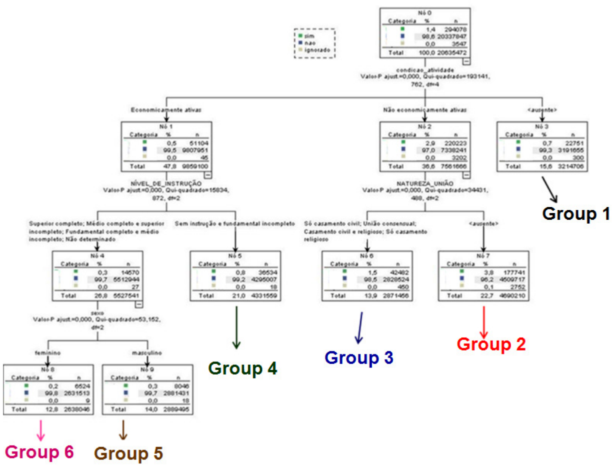

Fig. 4 Diagram AID dependent variable mental.

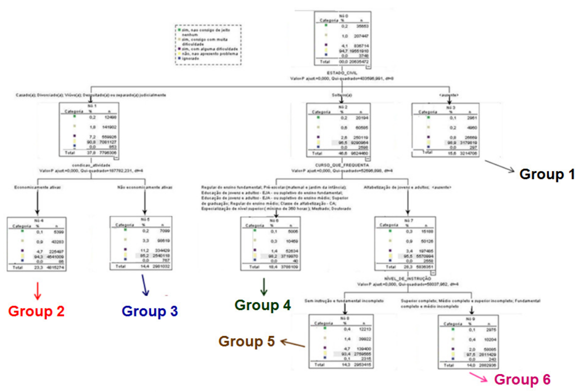

Fig. 5 Diagram AID dependent variable to listen.

already attended or attends public or private day care and returns home from work every day.

Among the different groups formed what best contributes to a lower incidence of disability for the group was around 4 formed by unmarried and attends public or private daycare and that contributes to the worsening of this deficiency was the group formed by three married state , divorced, widowed and separated or legally separated situation and not economically active.

In the case of Fig. 4, the variables that most contribute to a lower incidence of mental disabilities are economically active status, level of education from the complete primary onwards and female and the group formed that contributes to the higher incidence of mental problems was 4 formed by economically active status and education level of at most incomplete primary. Among the groups, that best contributes to the four, lower incidence of mental or intellectual is the group formed by six economically active status, level of education completed elementary or older and female.

Finally, in the case of Fig. 5, the variables that most contribute to a lower incidence of hearing impairment are unmarried, economically active condition, attends course from regular elementary and fundamental education level or more.

The group that best contributes to a lower incidence of hearing impairment is the group formed by four unmarried and attends regular course of school or more and profile that contributes to the increased incidence of hearing impairment is the group formed 3 by marital status married, widowed, divorced, separated or legally separated and not economically active situation followed by group 5 formed by unmarried, attends literacy course for young adults and educated incomplete primary or no education.

The variables that were selected for this study were provided activity in the five cases studied: marital status in four; education level in three; course and attends daycare in two, and finally, secondary job, returning home from work, sex and nature of the union are present in every single situation.

By doing an analysis of the plots in Figures 1-5 we can see that except mental disability (Figure 4), the first variable that partitions the other dependent

variables are different marital status, which in turn discriminates economically active status variable.

4. Conclusions

The greater the number of disabilities tends to be the lowest level of education and income that these people will get in their work.

The level yes, I can´t in any way tends to achieve better levels of education and income level rather than the level, but very difficult for variables to see, move and listen.

Note the existence of a large gap between people with disabilities and without disabilities as the level of education and income level, with the highest concentration of people with disabilities in the income ranges between 0 and 1 mw and between 1 and 3 mw, and; educated to uneducated incomplete primary. The higher the level of education and a higher income bracket, the greater the inequality between disabled and non-disabled.

The profile that most contributes to the incidence of these deficiencies is formed by being married, widowed, divorced, separated situation legally and economically inactive.

A deficiency in this population is the most visual and lower this is mental.

Deficiency that has the highest amount of people who can´t read and write is the mind while the one with the greatest amount of people who can read and write is visual.

Group of people who see, hear or move achieve results with some difficulty, although smaller, but very close to people who do not show any deficiency in terms of level of education and income.

The value of this research lies in the fact that there is virtually no work to do comparative study between people with disabilities and who do not have disabilities considering explanatory variables such as education, income, age and others.

As a continuation of this work in the future we propose:

First: new study comparing people who belong to groups that can not see, hear or move in any way they see and with some difficulty;

Second: Evaluate together with the possibility of improvements IBGE's census questionnaire adding the topic disability for each different disabilities a question that asks whether the disability is from birth or after birth was that age, justified by the fact people who acquired disability after a certain time the most difficulties to adapt this new condition which carries a certain time to adapt in their new condition;

Third: Application of new analysis using other techniques such as geostatistics, canonical correlation, structural equation modeling and other that may assist in the investigation between different disabilities and other variables considered as predictors;

Fourth: periodic repetition of the survey data in order to allow better monitoring of people with disabilities and allow other comparative studies can be made;

Fifth: Consider the relationship between the different disabilities and other variables collected in the census as a race, and other major work;

Sixth: Consider also the study by region, state and county, and finally;

Seventh: Submit new studies and monitoring of variables from the categories that contribute in enhancing the various deficiencies after using the technique AID as analysis.

The results of this work can benefit public managers in better support in the care of persons with disabilities to have knowledge of who they are, where they are and how.

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A Novel Method for Banks to Monitor the Cumulative Loss Due to Defaults

KSS Iyer

Abhijit Chirputkar, Yatin Jog and Vijayalakshmi. Symbiosis Institute of Telecom Management, Symbiosis International University, Pune, India

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Abstract: Banking institutions all over the world face significant challenge due to the cumulative loss due to defaults of borrowers of different types of loans. The cumulative default loss built up over a period of time could wipe out the capital cushion of the banks. The aim of this paper is to help the banks to forecast the cumulative loss and its volatility. Defaulting amounts are random and defaults occur at random instants of time. A non Markovian time dependent random point process is used to model the cumulative loss. The expected loss and volatility are evaluated analytically. They are functions of probability of default, probability of loss amount, recovery rate and time. Probability of default being the important contributor is evaluated using Hidden Markov modeling. Numerical results obtained validate the model.

Keywords: Random point process, expected cumulative loss, non Markovian, hidden Markov model.

1. Introduction

For most of the banks, loans are the largest and the most obvious source of credit risk. The banks should ensure adequate capital available against this risk. The higher the risk more chances of default and resultant losses to bank. Hence it is very important for banks to measure, monitor and control credit risk over a period of time. Banks all over the world face significant challenge due to default of borrowers. The turbulence that dominated financial markets throughout the world during the last three decades has proved the ineffectiveness of the standard risk management tools used. Most of these tools are based on the normal distribution and they do not account for the random events. Therefore a better modeling tool should be based on random point processes because defaults occur at random instants of time and defaulting amounts are random.

The cumulative loss due to defaults occurring at random instants of time and built up over a period of time could affect the income of banks and could reduce the capital cushion. Though large number of research publications dealing with default emphasize on Markovian approach, authors like Nickell, Peraudin and Varetto (2000), Bangia et al (2002) and Coudere F (2005) insist on including non Markovian features of the problem of cumulative loss. Agencies like Moodys and Standard & Poor have refined their initial ratings by adding certain variants to accommodate the non Markovian features of the problem. In this paper a time dependent non Markovian random point process is studied to model the expected cumulative loss and its volatility.

The aim of the study is to derive an compact expression for expected cumulative loss and volatility so that banks can use them with the data available. Random point process form a special class of stochastic process used in modeling wide range of problems with the following common characteristics.

Events occur at random instants of time and each event triggers a feature like random default loss. The

Corresponding authors: KSS Iyer, Symbiosis Institute of Telecom Management, Symbiosis International University. Email: kss_iyer@hotmail.com.

quantity that is of interest is the cumulative nature of the feature over a period of time. Stochastic processes associated with random points on a line were studied by Ramakrishnan (1953) and subsequently studied by S.K.Srinivasan and KSS Iyer (1970). In the present paper it is proposed to consider that each default occurs at random instant of time, not necessarily of Poisson type, triggers a random loss and the amount builds up over a period of time. The expected cumulative loss and its volatility are explicitly obtained and they are functions of probability of default, expected loss amount, recovery rate and time.

The paper is organized as follows: In section 2 random point process model is presented for the cumulative default loss. Section 3 considers the formulation of stochastic differential equation satisfied by the probability frequency function of cumulative loss. Using Fourier transform closed form analytic solutions are obtained for the first two moments of the cumulative loss for the Poisson arrival of defaults. For non Poissonian arrival a hidden Markov modeling is used to derive the probability of default in section 4. The maximum likelihood probability of default among borrowers of the same type of loan by classifying them into three states like least risk, medium risk and high risk. In the absence of real data the numerical results are obtained in terms of dimensionless ratio and the model is validated by the graphs.

2. Random Point Process Model

Consider defaults of type j occurring at random instants of time $t_1, t_2, \ldots t_n$. We associate with each t_i a random loss $a^j(t_i)$, where $a^j(t_i)$ are statistically independent random variables. We associate a time dependent deterministic hazard function $h^j(t, t_i)$ to capture the impact of the unit default loss of type j at t_i at a later time t where $t_i \leq t$. This could represent the built up rate of loss function over a period of time. For a typical realization of defaults at t_i for i = 1, 2, 3. . . N, the cumulative loss at time t may be written as

$$L^j(t) = \sum a^j(t_i).\ h^j(t, t_i) \text{ for } i = 1, 2, \qquad (1)$$

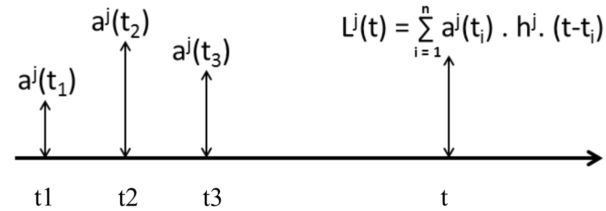


Fig. 1

It may be noted $L^j(t)$ is a random variable because $a^j(t_i)$ is random. Also $L^j(t)$ is non markovian and the normal markovian analysis cannot be carried out. However the probability frequency function for $L^j(t)$ can readily give the moments of $L^j(t)$.

$\sum L^j(t)$ for j = 1, 2, 3….k will give the total cumulative loss due to K types of loans. For the rest of the discussion, we drop the index j, as the solution for L(t) will be identical except for the default and recovery rates . The first moment of L(t) gives the expected cumulative loss at time t and second moment gives the volatility at time t.

3. Derivation of Stochastic Differential Equaton for L(t) for Poisson Arrival Rate of Defaults.

In this section we obtain the stochastic differential equation satisfied by $\pi(L,t)$ the probability frequency function governing the cumulative loss function L(t). We assume for the present discussion arrival of defaults obey Poisson distribution though the results obtained are equally valid for non Poissonian distribution for deterministic time dependent default rates. $\pi(L,t)dL$ denotes the probability that L takes a value between L and L+dL at time t assuming L to be 0 at time t = 0. With each default randomly occurring at t_i, we associate a loss $a(t_i).\ h.(t - t_i)$ where probability of random loss $a(t_i)$ is given by $p(a_i)$.

Analyzing the possible events in the time interval (0, Δt) there are two possible events

(1) A default occurs in (0, Δt) with probability $\lambda.\Delta t$ in which case a contribution of a.h(t) to L(t) arises and the corresponding contribution to $\pi(L, t)$ is given by the probability

$$\lambda\Delta t \,.\, \int_a \pi(L - a.h(t), t).\ p(a)\ e^{-\mu t}\ da$$

where the exponential term ensures that the probability of not coming out of default during the time t and μ represents the recovery rate. p(a) represents the probability of loss amount is a.

(2) No default occurs in $(0, \Delta t)$ with probability $(1 - \lambda\Delta t)$ in which case the corresponding contribution to $\pi(L, t)$ is

$$(1 - \lambda\Delta t).\ \pi(L, t\text{-}\Delta t)$$

Combining the above two we have partial differential equation for $\pi(L, t)$

$$\partial\pi(L,t)/\partial t = -\lambda\pi(L,t) + \lambda_a \int p(a).\ \pi(L - ah(t), t)\ e^{-\mu t}.\ da \qquad (3.1)$$

The above equation has been solved by using Fourier transform and by using the properties of characteristic function the moments of L(t) are derived.

Solution

Defining $\overline{\pi}(s,t) = \frac{1}{2\pi}\int_0^\infty \pi(L,t)\ e^{isL} dL$ equation 3.1 takes the form

$$\partial\overline{\pi}(s,t)/\partial t = -\lambda\overline{\pi}(s,t) + \int_0^\infty \lambda\overline{\pi}(s,t).p(a)e^{iash(t)}e^{-\mu t}\ da \qquad (3.2)$$

Equation (3.2) gives the characteristic function of L(t) viz $\pi(s,t)$.

This yields all the moments of L(t).

Defining $\varphi(u) = \int p(a)\ e^{iua} da$ and substituting in (3.2) we get

$$\partial\overline{\pi}(s,t)/\partial t = -\lambda\overline{\pi}(s,t) + \lambda\overline{\pi}(s,t)e^{-\mu t}\varphi(sh) \qquad (3.3)$$

The solution of the above equation can be written as

$$\overline{\pi}(s,t) = \exp\left[-\lambda t + \lambda\int_0^t e^{-\mu\tau}\ \Phi(sh)d\tau\right] \qquad (3.4)$$

By taking inverse Fourier transform of the above equation we can obtain $\pi(L,t)$.

$$\pi(L,t) = \frac{1}{2\pi}\int_0^\infty \exp[-isL + \sum\frac{(is)^n}{n!}\int_0^t \lambda h^n(T)e^{-\mu\tau}E[a^n(T)]dT]ds \qquad (3.5)$$

The n^{th} moment of the cumulative loss L(t) is given by

$$E[\,L(t)]^n = i^n.\partial^n/\partial s^n.\ \{\pi(s,t)\}|_{s=o} \qquad (3.6)$$

Equation (3.5) gives the probability of cumulative loss L at time t.

The expected cumulative loss and the volatility can be obtained from the first two moments of L(t). Differentiating Equation (3.3) gives the moments of L(t) and the first two moments are the first and second derivatives of π (s,t) with respect to s and for s = 0. Denoting N(t) and M(t) for the first two moments of L(t) we get the following equations satisfied by N(t) and M(t).

$$\partial N/\partial t = -\lambda N + \lambda N e^{-\mu t}\Phi(0) + h(t)\Phi'(0)\ \lambda e^{-\mu t} \qquad (3.7)$$

$$\partial M/\partial t = -\lambda M + \lambda e^{-\mu t}\,[M\Phi(0) + 2Nh(t)\ \Phi'(0) + h^2(t)\ \Phi''(0)] \qquad (3.8)$$

Solving (3.7) equation we get,

$$N(t) = h\ \Phi'(0)\ \lambda e^{-\Psi(t)}\int_0^t e^{\Psi(T)-\mu T}dT \qquad (3.9)$$

Where

$\Psi(t) = \lambda t - \lambda\Phi(0)/\mu(1 - e^{-\mu t})$

$\Phi(0)=p_0(a_m\text{-}a_0)$

$\Phi'(0)= p_0\ (a_m^2\text{-}a_0^2)/2$

a_m=Maximum Loss amount

a_0=Minimum Loss amount

Solving (3.8) equation we get,

$$M(t) = \int_0^t {}^{t}\, e^{\lambda\ \mu\ t2/2}\ \Phi(0).\ \lambda e^{-\ \mu t}[2N(t)\ h^2(t)\ \Phi'(0) + h^2(t)\ \Phi''(0)]dt \qquad (3.10)$$

M(t) gives the second moment and the volatility of the cumulative loss can be evaluated from M(t) and N(t). It may be noted it depends on λ, μ, a_m, a_0, p_0 and h(t) the built up rate of loan over a period of time t.

The expected cumulative loss depends on probability of default λ, recovery rate μ, built up rate of defaulted amount h and p_0 the probability of the loss amount. The probability of default is an important measure for banks. In the next section, we discuss the hidden markov modeling approach to evaluate the

probability of default.

4. Hidden Markov Modeling to Evaluate Probability of Default

Hidden Markov models have been used in variety of applications like Speech Recognition, signal processing, artificial intelligence, computational biology, finance, image processing and bio statistics. These models possess the ability to present unobservable developments that take place with observable out puts. A comprehensive treatment of Hidden Markov models can be found in Rabiner (1989).

In this section we propose to evaluate probability of default using hidden markov model. Hidden Markov modeling consists of two sets of states namely hidden states and observable states. Hidden states are described by Markov Process and their transition probabilities are given by transition matrix.

The observed output stochastically depends on hidden states. Emission probabilities from hidden states to observable states are given by the emission matrix. The initial probabilities for the hidden states are known.

Elements of a discrete HMM are:

(1) Hidden states $Q = [\ q_i\ ]$, $i = 1, 2,\ldots N$

(2) Transition probabilities $A = \{\ a_{ij} = P(q_j)$at $t+1/ P(q_i)$ at $t\}$, where $P(a/b)$ is the conditional probability of a given b and $t = 1, 2, \ldots T$.

(3) Observations $O = \{o_k\ \}$, $k= 1,2,\ldots M$

(4) Emission probabilities $B = \{\ b_{ik} = b_i(o_k) = P(o_k/q_i)\ \}$. B is the probability that the output is o_k given the current state is q_i.

(5) Initial state probabilities $\pi = \{\ p_i = P(\ q_i$ at $t = 1)\ \}$

The model is characterized by the complete set of parameters $\lambda = \{\ A, B, \pi\ \}$

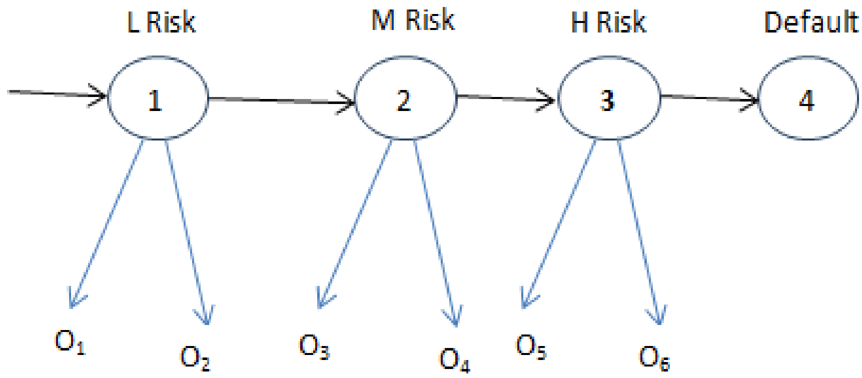
The three important problems that can be solved using HMM are: (a) given the model the computation of the probability of the occurrence of the observation sequence (b) given the model the sequence of the states that maximizes the joint probability of state sequences and observations sequence (c) adjustment of the parameters of the model to maximize either of the above two probabilities.

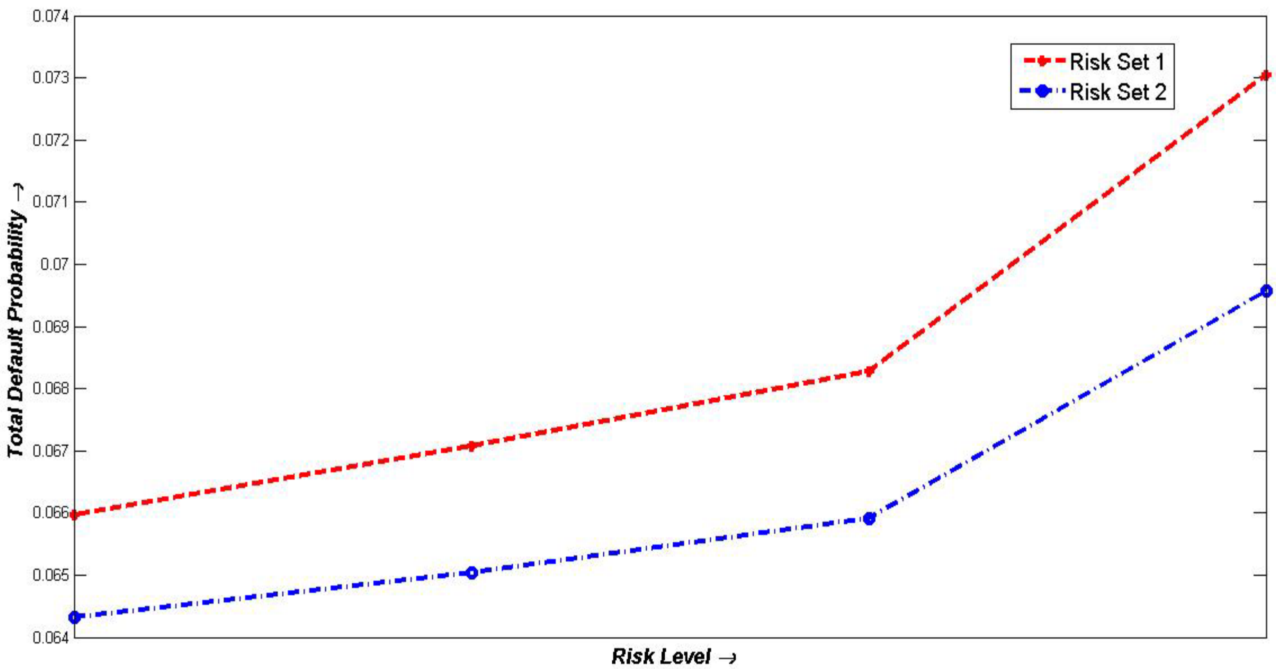
5. Working Methodology for Calculating Probability of Default

We propose to consider the population of borrowers of type j loan in three groups like least risk, medium risk and high risk based on the risk levels from past data of the borrowers available with banks. We consider the number of states as four including the default state. State 1 is least risky, state 2 is medium risk, state 3 is high risk and state 4 is default state or the absorbing state. The first three states generate the observable states namely risk levels and the state transition takes place from left to right among the states with default as absorbing state as shown in Fig. 2.

We assume that default can come from any of the three groups of borrowers with three different risk levels. Default is the absorbing state and the best sequence is the one that maximizes the probability for the given observation sequence ending with default state. For simplicity it is assumed that the transitions among hidden states are considered from left to right. In other words the transition matrix is not Ergodic. Viterbi's algorithm chooses the best state sequence that maximizes the likely hood of the state sequence for given observation sequence when the Markov process is not ergodic. For ergodic matrix posterior decoding can be used.

Fig. 3 depicts numerical results from Viterbi decoding of the hidden markov model of Figure 2. The plot shows variation in total probability of default with increasing risk levels of the low, medium and high risk states. The levels of risk are implemented as the emission probabilities of the observable states O_1-O_6. Two sets of risk levels have been studied to ascertain the variation in default probability with risk. From the plot it is clear that the probability of default increases gradually with increasing risk levels of the various states in the model.





Numerical Results: In the absence of real data, to gain some insight into the validity of the model, we have plotted the expected cumulative loss in terms of a non-dimensional ratio of maximum loss to minimum loss for default. N(t) is the expected cumulative loss and is function of λ,μ,h(t) and p(a). Assuming h(t) and p(a) as constants h and p_0 we get ,

$N(t) = \lambda p_0 h(r^2-1)/2\ .\ (\ (a_0^2[e^{-\mu t} - e^{-\lambda t(1-\Phi_0)}])\ /\ ([\ \lambda(1-\Phi(0))-\mu])\ )$

Where $r = a_m/a_0$

The above expression can be written to the first approximation in a non-dimensional form

$N(t) / (\lambda p_0\ h\ a_0^2) = (\ (r^2-1)/2\ )\ .\ t$

Fig. 4 represents the above graph for different values of t and r.

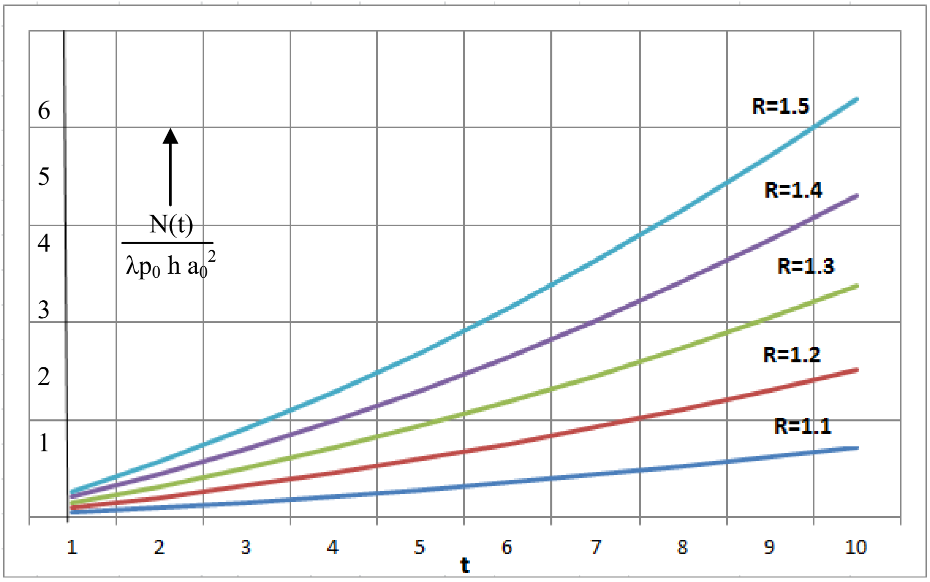
The graph indicates at t=0 the cumulative loss is zero for all values of r. Also when r =1 namely $a_m = a_0$ the cumulative loss is zero. The cumulative loss increases with increasing time and values of r.

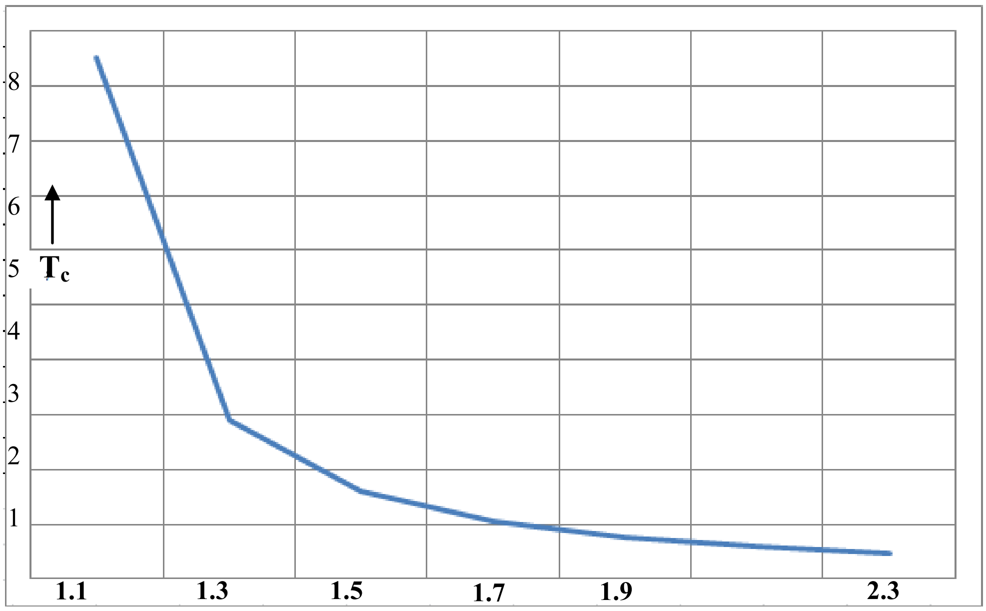
Time required for cumulative loss to reach critical value.

Assuming the cumulative loss N(t) reaches the maximum value N_c after a time T_c. we have plotted T_c for different values of from the following expression

$T_c = (\ (2N_c)/(r^2-1)\ )\ .\ (\ 1\ /\ (\lambda p_0\ h\ a_0^2.1)\ )$

From the graph in Fig. 5 it is seen that the time taken to reach critical cumulative loss falls rapidly with increasing values of r and the curve is asymptotic for r = 1.





6. Conclusion

In this paper an attempt is made to model the cumulative default loss for different types of loans as a random point process with defaults occurring at random instants of time. The expected cumulative loss for a homogeneous type of loan at any time is expressed in terms of probability of default, recovery rate, built up rate of loss amount over a period of time and the probability of loss amount. The total sum of the cumulative loss due to all types of loan can be evaluated from the expression $\sum E\{L^j(t)\}$ for different values of j = 1, 2, ... etc. We presume banks can calculate the above expression from the data available to them. The time dependent cumulative default loss is plotted against the non-dimensional ratio of maximum loss to minimum loss for default in the absence of real data. The time to reach the critical default loss is also plotted. The expression for the second moment of the cumulative loss, which will give the volatility, is also evaluated. Results obtained in this paper are equally valid in more general case when the probability of default depends on time. It is expected that the analytic solution derived in this paper could provide a rational

basis for banks to forecast the time to cumulative default loss that could wipe out the capital.

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Impact and Benefits of Micro-Databases' Integration on the Statistics of the *Banco de Portugal*

Paula Menezes, Luís D'Aguiar
Statistics Department, Banco de Portugal, Portugal

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Abstract: Data are critically important to good decision-making. In an increasingly complex economy, conventional data collecting schemes are no longer sufficient. To deal with the challenge of maintaining its statistics relevant to the users in an ever-shifting environment, the *Banco de Portugal* decided to explore the largely unused statistical potential of the available micro-databases and to integrate the existing administrative and survey data, thus enhancing the basic information infrastructure while protecting confidentiality. This presentation will address the benefits and problems to be dealt with when two or more data-sources are to be integrated.

Key words: Micro-data, infrastructure, integration, knowledge

1. Introduction

Economies are constantly faced with new challenges. To remain relevant, official statistics have to keep up with the rapid changes of modern times, which typically require the availability of commensurate statistical data that users may exploit in an accurate and reliable way. Policy-makers, financial supervisors and regulators, just to name a few, require as much rich and timely information as possible to take appropriate decisions.

The *Banco de Portugal* (hereinafter referred as "the Bank") – or any other major producer of official statistics, for that matter – has to ensure that the statistics for which the Bank is accountable retain relevancy over time and are able to cope with the speed and the scope of the main stakeholders' ever-increasing demand for comprehensive, detailed and high-quality information.

However, the process of continuously adapting the statistical output to new phenomena has a number of serious limitations. Conventional data collecting systems cannot simply keep on expanding indefinitely to cope with the ever-increasing need to fill the information gaps perceived by the users or in anticipation to their possible future data requirements. Amongst the possible motives for not pursuing recurrently this approach one could point out, *inter alia*, the following:

- The resulting overburdening of respondents goes against well-established best practices.
- The related initial and maintenance costs are far from being negligible, both to the agency that collects the data and to the respondents.
- New statistical datasets (or significant enhancements to existing ones) require lengthy preparation time (years, rather than months) and, once launched, are supposed to remain in operation for a prolonged period of time (typically around five years, in the case of the Eurosystem statistical reporting systems). This time-lag could even be further extended, should the revision result from a major methodological change, as it is often the case.
- *Ad hoc* surveys are, in general, too time-consuming and expensive, not to mention reliant upon the willingness to participate on the part of the

Corresponding authors: Paula Menezes, Statistics Department, Banco de Portugal, PORTUGAL. Email: pamenezes@bportugal.pt.

target population.

In fact, the response given by conventional data collecting systems to new statistical demands – stemming from, *e.g.*, the need to conduct macro-prudential analyses or to accommodate new data requirements related to the Bank's participation in the European System of Central Banks (ESCB) – is problematic, costly and could possibly turn out to be counterproductive, which helps to understand why more and more central banks have been reusing the available micro-data, thus recognising that such information is both useful and necessary to respond to the data requirements of the complex world we live in, and to better address new issues and challenges as they arise.

Throughout this paper the term "micro-data" will be used to refer data about individual persons, households, businesses or other entities; it may be data directly collected by the Bank or obtained from other sources, such as administrative sources.

2. From Direct Integration to Integration via a Data Warehouse

2.1 Changing the Statistical Paradigm

Data are critically important in making well-informed decisions. Poor quality data or, *a fortiori*, lack of data can lead to an inefficient allocation of resources and imposes high costs on the society. In ever more complex economies, the traditional approach to the compilation of official statistics – *i.e.*, producing standard statistical tables that can only address a set of predefined questions – is becoming increasingly insufficient and ineffective.

The Bank's strategy to deal with the challenge of maintaining its statistics relevant to the users in a shifting and more demanding environment, while attending to the need to keep the reporting burden on respondents at an acceptable level, was to enhance the overall efficiency of the statistical framework by further exploring the largely unused statistical potential of already existing data sources. In fact, statistically edited micro-data, which include *e.g.* data from administrative sources not originally intended for statistical purposes or even data related to the Bank's prudential supervision function, offer an unusual array of interesting features, *inter alia*:

- *Very good coverage* of the population in most of the cases.
- *Relatively low reporting costs*, thus helping to mitigate the constraints imposed by the response burden of the reporting agents.
- *Increased flexibility and agility* as regards the compilation of new statistics, *e.g.* related to financial and other structural innovations.
- *More rapid response to* ad hoc *data requirements* from the users – in many cases, almost in real time.

Moreover, the evolution in network and communication protocols, database systems and multidimensional analytical systems has somewhat removed the potential disadvantages of having to deal with the huge amounts of data normally associated with the handling of micro-databases. (Aguiar *et al.*, 2011)

Best practices in compiling official statistics advocate that all data should be collected only once: any form of double reporting or redundant collection should be avoided and, if existing, be terminated. Accordingly, data already available – due to whatever reasons – should be reused, if found useful, for statistical purposes. Obvious candidates are data from existing Central Credit Registers, as well as data from Central Balance Sheet Offices databases and information collected within the framework of the Bank's prudential supervision function. The experience of the Bank in this area has shown that the use of such information for statistical purposes can lead to a significant reduction of the response burden, higher data quality and lower costs.

On the national level, a formal exchange of administrative data with institutions outside the central bank, like the national statistical institute (NSI) or the tax authorities, would also help to reduce the reporting costs. An important precondition would be the

maintenance of common company registers with the NSI. Extending this idea across national borders, one could think of common international databases – *e.g.*, exchanging micro-data on significant cross-border mergers and acquisitions that need to be recorded symmetrically in the respective statistics of both affected countries. (Liebscher *et al.*, 2008)

2.2 Micro-Databases Managed by the Banco de Portugal

For the last 10 years the Bank has been developing and maintaining several micro-databases based on item-by-item reporting and has been exploring the statistical potential of these complementary sources of information with significant positive impacts on the overall quality of its statistical output.

The databases managed by the Bank's Statistics Department include:

- The *Securities Statistics Integrated System* (SSIS) database, a security-by-security and an investor-by-investor database that provides, in a single repository, data on the securities issues and holdings required by the different statistical domains (*e.g.*, monetary and financial statistics, external statistics, securities statistics and financial accounts), thus replacing the separate and distinctive data storing systems that were previously in place.
- The *Central Credit Register* (CCR), an administrative database that stores credit-related information supplied by all the resident credit-granting financial institutions.
- The *Central Balance Sheet Database* (CBSD), which stores granular information on virtually all the resident corporations, collected through the so-called *Informação Empresarial Simplificada* (IES), a joint effort of four distinct Portuguese public entities – the Ministry of Finance, the Ministry of Justice, *Instituto Nacional de Estatística* (the Portuguese NSI) and the *Banco de Portugal* – consisting of yearly submissions of information by corporations, in a single, paper-free, electronic form, to fulfil reporting obligations of accounting, fiscal and statistical nature.

Besides complementing and helping to cross-check the information gathered through the conventional channels, these micro-data have proved to be of great importance to the understanding of the developments in the Portuguese financial system, especially in the wake of the recent global financial crisis.

So far, this approach has permitted, *inter alia*:

- *Improving the responsiveness to new users' requirements*, particularly those arising from *ad hoc* information requests, with proven results in reducing or eliminating data gaps and in monitoring and assessing the evolution of the Portuguese financial system.
- *Curtailing the follow-up procedures as regards data collecting schemes*, whereby respondents are re-contacted after the initial submission of data, to obtain missing information and/or to verify and, if necessary, to correct questionable data.
- *Enhancing the quality control procedures* (*e.g.*, by cross-checking elementary/raw data from different statistical domains), thus increasing the efficiency of the production process and improving the quality of end-products.
- *Avoiding data redundancy*, while at the same time expanding significantly the range of statistics available.

As an example, the use of the available micro-databases for the compilation of the Portuguese flow-of-funds within the national financial accounts has been extremely helpful, as it allows for a much better understanding of the interlinks within the resident economy and *vis-à-vis* the rest-of-the-world.

2.3 Deepening Data Integration

In keeping with such course of action, the Bank has been developing an approach that, once completed, will allow for a higher level of integration of the available administrative and survey data. The goal is to achieve a significant enhancement of the basic data infrastructure without jeopardizing the provisions in the legislation, codes of practice and protocols that

protect data confidentiality. In addition, a reduction in respondent burden and an increase in the breadth and depth of the information available to policy-makers and researchers are expected.

An architecture based on business intelligence – broadly defined as a category of applications (e.g., decision support systems, query and reporting, online analytical processing, statistical analysis, forecasting, and data mining) and technologies for gathering, storing, analyzing, and providing access to data, to help a variety of users to make better business decisions (Terzić, 2008) – could significantly contribute to meeting the Bank's concerns in this area. With this in mind, the Bank set off a study in 2008 aiming at defining a business intelligence framework to be used as a reference in all future information technology developments in the statistical realm. This framework will be built upon three pillars (Aguiar et al., 2011):

- A common technological infrastructure across the various information systems, to facilitate the integration and re-usage of components and to promote data access efficiency and transparency to final users.
- A centralised reference database, to provide common reference data (e.g., identification criteria for the relevant entities that are observed, characterization of variables and classifications) and to enable the linkage of information across different sources and systems.
- A data warehouse approach, to guarantee a central access point to all statistical data, independently of the input source or the production process. This implies, inter alia, a data structure specified on the basis of common criteria, valid across the different sets of data.

At the moment, the Bank's statistical information subsystems are in the process of being reformulated according to this model: the SSIS and the balance of payment and international investment position statistics, on one hand; the CBSD and the CCR, on the other hand.

Data integration is concerned with integrating unit record data from different administrative and/or survey

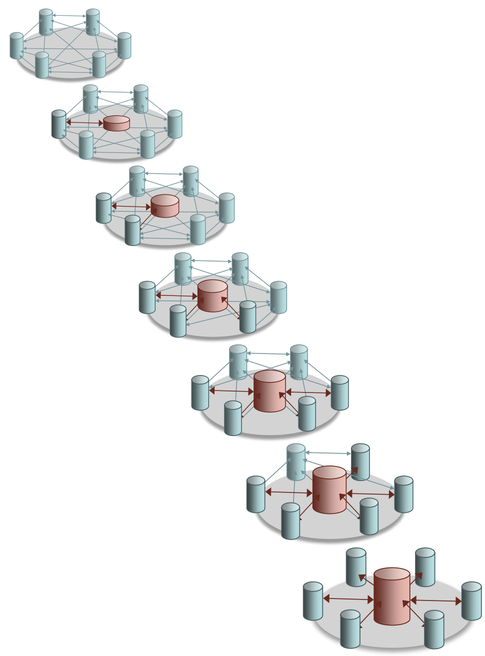

Fig.1 From direct integration to integration via a data warehouse.

sources to compile new official statistics which can then be released in their own right. Integration of micro-data is a powerful approach to enrich the already available information – *e.g.*, by allowing efficient cross-data comparisons and quality checks among the different statistical domains. Surely, this is easier said than done – it is a rather complex process and, pending on the degree of integration to be achieved, it can be characterized by different features.

The prevailing, stand-alone, islands of information may be very diverse, making it technically difficult to create homogeneous information systems. In addition, there are many different *practical and methodological problems that must be previously addressed* when two or more sources are to be integrated, *inter alia* (Di Zio, 1998):

- Harmonising populations – *e.g.* determining the group of entities that belong to a given institutional sector (financial and non-financial corporations, general government, households and non-profit institutions serving households) –, identification criteria, reference periods (annually, quarterly,...), variables and classifications.

- Adjusting for measurement errors (accuracy) and for missing data.
- Deriving variables.

However, such shortcomings may very well be offset by the possible *benefits of integrated data sets*. The latter include, according to UNECE (2009):

- Compiling new or enhanced statistics.
- Producing more disaggregated information for measures where some information currently exists.
- Carrying out research using composite micro-data that cover a wider range of variables for a larger number of units than available from any single data source.
- Potentially improving or validating existing data sources.
- Possibly reducing respondent burden.

These benefits could be illustrated by the following case, extracted from the Bank's own statistics: a given corporation, providing annual accounting data under its IES reporting obligations, might also be answering to the Bank's ISII survey (*Inquérito sobre Investimento Internacional*) and, at the same time, having its securities issues and holdings recorded in the SSIS database; in an integrated system, it would be possible to ensure the compatibility of these data at a micro-level, thus providing a powerful tool for the compilation of financial accounts (which require that total uses equal total resources in the domestic economy). Nonetheless, as referred above, having a partial integration, *e.g.* one that allows for a unified view on two different sub-systems like the CBSD and the CCR, clearly enriches analytical data awareness. In fact, the idea of pooling together all the data on financial or non-financial corporations available at the *Banco de Portugal* is rather appealing; it would allow us to have a more specific and detailed view on this particular institutional sector. In the case of the financial corporations, the advantages would be even stronger should we take into account the information of yet another important subsystem: the data used for supervisory purposes.

4. Concluding Remarks

The implementation of an architecture framework such as the one summarized above will contribute to the construction of a coherent and integrated statistical system as opposed to having multiple systems that coexist but are not connected in an efficient way.

Such approach has only been possible because of the possibilities brought in by the information technology (IT) revolution. But even though IT has enabled the statistical community to carry out the current procedures for collecting, compiling and disseminating statistics more efficiently, albeit at a non-negligible cost, it is important to reflect on how such revolution can be used to introduce new and more effective procedures.

Benefits are evident but there are also problems, challenges and cautions with the use of integrated micro-data, particularly those related to confidentiality issues. As said before, data already available should be reused if found useful for (other) statistical purposes; that being the case, it is necessary to strictly safeguard their confidentiality and to ensure that the sharing is legally allowed or explicitly agreed by the reporting agents. However, because of legal constraints, confidentiality makes the access to some useful data sources problematic and disclosure is a constant problem when we need to release data.

A data integration process is complex and can be characterized by different steps. One of these steps is adopting a unified view on the existing micro-data data sources creating a customized view on a sub-set of data (*e.g.* the financial or non-financial sectors).

Integrated micro-data have the potential to support, if need be, the drilling down of the most summarized levels of data to the most detailed ones, which may help to confirm (or to disprove) trends and developments conveyed by macroeconomic statistics and, concomitantly, to explore and/or to elucidate their possible implications for *e.g.* financial stability analysis and systemic risk assessment. (D'Aguiar *et al.*, 2011).

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Using Sobel Operator for Automatic Edge Detection in Medical Images

A.V. Doronicheva, A. A. Socolov and S. Z. Savin

Computing CenterFEB RAS Russia, 680063, Khabarovsk, Kim Yu Chen 65, Rusia

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Abstract: The problems of installation and integration of complex suite of software for processing medical images. Based analysis of the situation is realized in an easier integration of an automated system using the latest information technologies using the web - environment for analysis and segmentation of DICOM - images.

Key words: Segmentation, medical image, the algorithm contouring, the Ring Road, the quality of medical images

1. Introduction

The object to identify features, to compare them with the data from the library and infer the presence probability of anomalies, to preallocate to a plurality of present on a specific image. In most cases, the investigated images are present noise, the texture similar distortion to those regions owned by the object under investigation. All this complicates process of selecting objects and correctly display their borders, so contouring and segmentation algorithms have a very important role in the automated processing.

The operator Sobel - this is one the best algorithms available border selection, It is often applied as one of steps a more difficult and exact algorithms (such as an operator Kenny).

2. Experimental Section

The Sobel operator used in image processing. Often it is used in the algorithms border selection. This is a discrete differential operator that calculates the approximate value the image brightness gradient. The result of applying Sobel operator at each point in the image is a vector of brightness gradient at this point, or a norm. [1]

Easier to understand, the operator computes the gradient of the image brightness at each point. So is the direction of maximum increase brightness and the change value in this direction. The result shows how "sharply" or "smoothly" the image changes brightness at each point, and thus the finding a point probability on the face and the boundary orientation. In practice, the change amount calculation the of of brightness (its probability of belonging to a face) reliably and simpler to interpret than the direction the calculation. [2]

Mathematically, the a function gradient of two variables for each point in the image (which is a brightness function) - a two-dimensional vector whose components derivatives are the brightness of the image horizontally and vertically. In each point of the image gradient vector is oriented in the maximum increase brightness direction, and its length corresponds to the change in brightness. This means that the Sobel operator result at the constant brightness point is the zero vector, and at a point on the border different brightness levels areas - vector crossing the border in the increasing brightness direction. [3]

Segmentation process using the Sobel operator is based on a simple filter mask moving from point to point in the image, at each point (x, y) filter response is calculated using the originally-formed bonds. In case

Corresponding authors: A.V. Doronicheva, Computing CenterFEB RAS Russia. Email: dream_fox@list.ru.

the linear spatial filter response given by the sum of multiplying filter coefficients the corresponding pixel value in the region covered by the mask filter. For 3×3 mask element illustrated in Fig. 1, the result

$$R = w(-1,-1)f(x-1,y-1) + w(-1,0)f(x-1,y) + ... + w(0,0)f(x,y) + ... + w(1,0)f(x+1,y) + + w(1,1)f(x+1,y+1) \quad (1.1)$$

that, apparently, is the sum of the multiplying coefficient mask and pixel values directly under the mask. In particular, note that the coefficient w (0, 0) is at a value of $f(x, y)$, thereby indicating that the mask is centered at the point (x, y). Upon detection of brightness differences can be discrete analogues of first and second order.

The first derivative dimensional function f (x) is defined as the difference between neighboring elements values:

$$\frac{\partial f}{\partial x} = f(x+1) - f(x) \quad (1.2)$$

Here we used the recording as a partial derivative in order to keep the same notation as in the two variables $f(x, y)$ case, where it is necessary to deal with the partial derivatives of the two spatial axes.

Similarly, the second derivative is defined as the difference between adjacent values of the first derivative:

$$\frac{\partial^2 f}{\partial x^2} = f(x+1) + f(x-1) - 2f(x) \quad (1.3)$$

The the first derivative calculation of the digital image is based on different discrete approximations of the two-dimensional gradient. By definition, the image gradient $f(x, y)$ at point (x, y) - a vector:

$$\nabla f = \begin{bmatrix} Gx \\ Gy \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} \quad (1.4)$$

As is known from mathematical analysis, the gradient vector coincides direction with the maximum rate direction of change function f at the point (x, y).

An important role is has detection the contours this

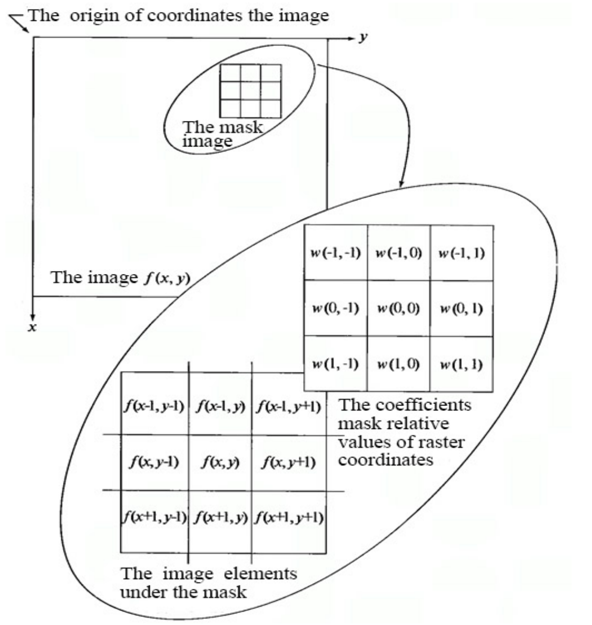

Fig. 1 The spatial filtering scheme. (response) R linear filtering point (x, y) images is:

vector module, denoted ∇f and is equal

$$\nabla f = |\nabla f| = \sqrt{G_X^2 + G_Y^2} \quad (1.5)$$

This value is equal to the change function f maximum rate at the point (x, y).

The gradient vector direction is also an important characteristic. Denote $\alpha(x, y)$ the angle between the vector ∇f at the point (x, y) and the axis x. As is known from mathematical analysis

$$\alpha(x, y) = arctg\left(\frac{Gy}{Gx}\right) \quad (1.6)$$

It is easy to find the contour direction in the point (x, y), which is perpendicular to the gradient vector at this point. And you can calculate the gradient of the image, calculating the value of the partial derivatives $\partial f / \partial x$ and $\partial f / \partial y$ for each point.

G_x and G_y - these are two of the matrix, where each point contains the approximate derivatives in x and y. They are calculated by multiplying the matrices G_x and G_y, and the summation of both matrices, as a result of the result is written to the current x and y coordinates in a new image:

$$G = \sqrt{G_X^2 + G_Y^2} \quad (1.7)$$

Matrix Gy and Gx:

$$G_Y = \begin{bmatrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ +1 & +2 & +1 \end{bmatrix} \times A \qquad (1.8)$$

and

$$G_x = \begin{bmatrix} -1 & 0 & +1 \\ -2 & 0 & +2 \\ -1 & 0 & +1 \end{bmatrix} \times A \qquad (1.9)$$

3. Results and Discussion

The result of applying Sobel operator is a two-dimensional the gradient map for each point. This can be process and display a picture, which sections to a gradient large value (mainly faces) will appear as white lines. [4, 5] The following images illustrate this with an segmentation example of medical images:

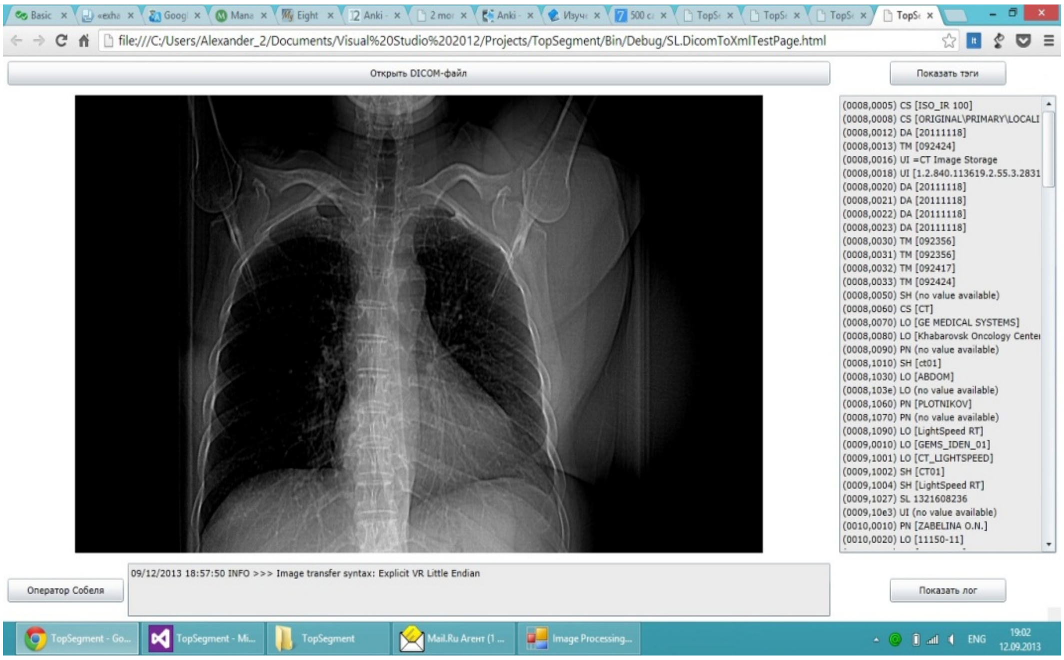

Fig. 2 The initial image without using Sobel operator.

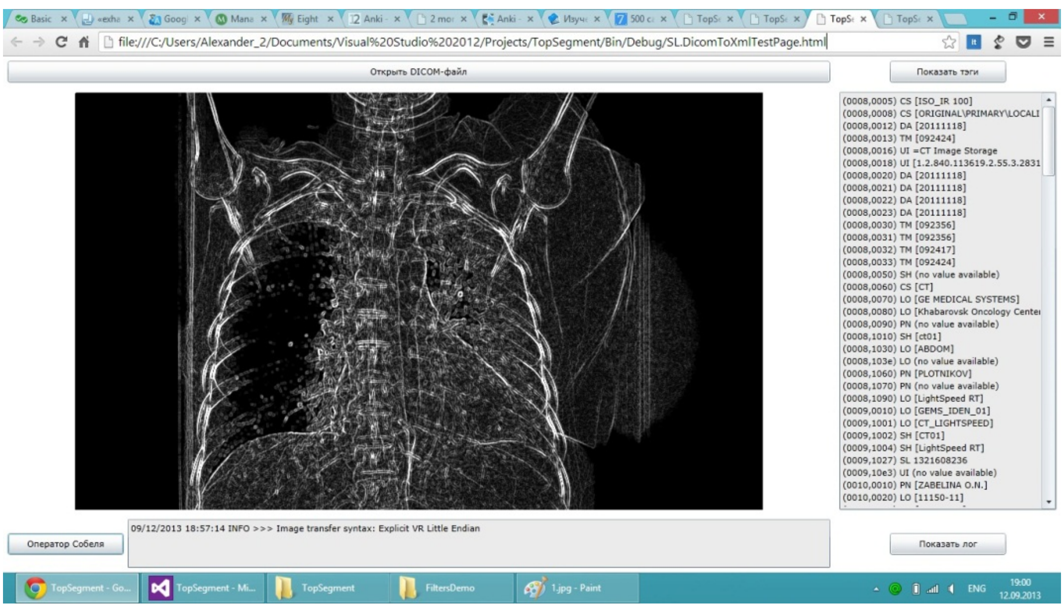

Fig. 3. Image using Sobel operator.

4. Conclusions

Thereby demonstrated the practical application of an experimental Sobel algorithm to analyze medical images. Therefore using Sobel operator in computed automated system allows greatly increase diagnostic possibilities of medical images segmentation for osteoscintigraphy analysis.

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On Pairwise Singular Sets and Pairwise Singular Maps

Rina Pankaj Verma[1], Mamta Singh[2]

1. School of Studies in Mathematics, Vikram University Ujjain (M.P.), India

2. Department of Appl. Sci. & Comp. Tech. Bundelkhand University, Jhansi, India

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Abstract: In this paper we introduce the concept of pairwise singular sets and pairwise singular maps between pairwise locally compact and pairwise hausdorff spaces and study the properties of pairwise singular maps.

Keywords: Pairwise singular sets, pairwise singular maps

1. Introduction

A *Bitopological* space is a triple $(X, \mathfrak{I}_1, \mathfrak{I}_2)$, where $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are topologies on a set X. J. C. Kelly [11] initiated the systematic study of such spaces and several other authors namely Weston [15], lane [6], Patty etc. contributed to the development of the theory. Kelly introduced pairwise Hausdorff spaces, pairwise regular spaces and pairwise normal spaces in the theory [11].

Recall that a cover U of $(X, \mathfrak{I}_1, \mathfrak{I}_2)$ is called *pairwise open* if $U \subset \mathfrak{I}_1 \cup \mathfrak{I}_2$. and U contains at least one non-empty member of $\mathfrak{I}_1$ and one non empty member of $\mathfrak{I}_2$. If every pairwise open cover of $(X, \mathfrak{I}_1, \mathfrak{I}_2)$ has a finite sub cover then the space is called *pairwise compact* [5].

According to I. l. Reilly [9] a bitopogical space $(X, \mathfrak{I}_1, \mathfrak{I}_2)$ is called *pairwise locally compact space* if $\mathfrak{I}_1$ is locally compact with respect to $\mathfrak{I}_1$ and $\mathfrak{I}_2$ is locally compact with respect to $\mathfrak{I}_1$. $\mathfrak{I}_1$ is locally compact with respect to $\mathfrak{I}_2$. if each point of X has a $\mathfrak{I}_1$ open neighbourhood whose $\mathfrak{I}_2$. closure is pairwise compact.

A space $(X, \mathfrak{I}_1, \mathfrak{I}_2)$ is called *pairwise Hausdorff* if for two distinct points x and y there is a $\mathfrak{I}_1$ neighbourhood U of x and a $\mathfrak{I}_2$ neighbourhood V of y such that $U \cap V = \phi$ [11] .

A function f: $(X, \mathfrak{I}_1, \mathfrak{I}_2) \to (Y, \mathcal{L}_1, \mathcal{L}_2)$ is called *pairwise continuous* if the induced functions $f : (X, \mathfrak{I}_1) \to (Y, \mathcal{L}_1)$ and $f : (X, \mathfrak{I}_2) \to (Y, \mathcal{L}_2)$ are continuous.

The notion of the singular set of a mapping defined by Whyburn and Cain [2, 14] was further investigated by various workers including Cain, Chandler, Tzung, Magill, Jr. Faulkner and Duda etc.(1981) [2, 3, 5]. Different workers described the concept with their own point of view. later this concept led to the notion of a singular compactification. Infact the compactifications which are known as singular compactifications are instances of Whyburn's unified space. Various authors have seen methods of constructing a compactification of X. Given a map f : $X \to Y$ where Y is compact authors who should be mentioned are Steiner and Steiner, Magill, Blakley, Gerlits and Magill, and Chou. Chandler and Tzung defined the remainder induced by $f : X \to Y$ to be $L(f) = \cap \{Cl_Y(X - F) \mid F \in K_X\}$ and proved that whenever Y is compact there is a compactification αX with $\alpha X - X$ homeomorphic to $L(f)$. Cain defined the singular set of a map $f : X \to Y$ in the following manner.

$$S(f) = \{p \in Y / \forall U \in \eta(p),$$
$$\exists F \in K_X, p \in F \subseteq U \text{ and } f^{-1}(F) \notin K_X\}.$$

Chandler and Tzung defined the remainder induced by f to be

$L(f) = \cap\{Cl_Y f(X - F) / F \in K_X\} = \{p \in Y / \forall U \in$

Corresponding authors: Rina Pankaj Verma, School of Studies in Mathematics, Vikram University Ujjain (M.P.). Email: rinaverma1981@yahoo.co.in.

$\eta(p)$, $\forall$ F $\in K_X$, $U \cap f(X - F) \neq \phi\}$

Other definitions of singular set given by Whyburn, Chandler, Faulkner etc. of a map between locally compact spaces are respectively given as

$$S'(f) = \{p \in Y \,/\, \forall\, U \in \eta_K(P), f^{-1}(\overline{U}) \notin K_X\}$$

and

$$\overline{S(f)} = \{p \in Y \,/\, \forall\, U \in \eta_K(P), \overline{f^{-1}(U)} \notin K_X\}.$$

Further it was observed in a paper Singular sets and remainders [2] for locally compact and Hausdorff spaces that all the definitions taken by various workers are equivalent and each of the sets described above are actually the same.

A *G-space* is a Hausdorff space X on which a topological group G acts continuously. The orbit space of a G-space X is denoted by X/G and the maps which assigns to x in X its orbit is the *orbit map* on X. An *equivariant map* f from a G-space X to a G-space Y induces a continuous map f_G on orbit spaces which sends the orbit of x in X to the orbit of f(x). Let H be a compact subgroup of G and let X be an H-space then for $h \in H$ and $(g, x) \in G\times X$, $h(gx) = (gh^{-1}, hx)$ defines an action of H on G×X. The orbit space $G \times_H X$ of the H-space G×X is called the twisted product of G and X [13].

For H-spaces X and Y, and equivariant map $f: X \to Y$ determines a continuous map $f_H: G \times_H X \to G \times_H Y$ which sends $[g, X]$ in $G \times_H X$ to $[g, f(x)]$ in $G \times_H Y$.

We begin with the study of pairwise singular maps and pairwise singular sets and see different definitions of pairwise singular set which are equivalent in section 2. A brief account of properties of pairwise singular maps in relation to product, composition, restriction of continuous maps etc. is given in section 3. The last section 4 is devoted to see the behaviour of pairwise singular maps in relation to various structures like G-spaces etc.

For terms and notations not explained here we refer the readers to [1, 7, 10, 13].

2. Pairwise Singular Sets

In this section the notion of the singular set of a map f between pairwise locally compact, pairwise Hausdorff bitopological spaces is introduced. It is proved that like topological spaces the four sets

$S_B(f)$, $S_B'(f)$, $\overline{S_B(f)}$ and $\mathcal{L}_B(f)$ are the same.

2.1 Definition

Let f: $(X, \Im_1, \Im_2) \to (Y, \mathcal{L}_1, \mathcal{L}_2)$ be a pairwise continuous map where X, Y are pairwise locally compact spaces. Then a point $y \in Y$ is called $\mathcal{L}_1$ *singular point* with respect to $\Im_2$ if for each open set $U \in \mathcal{L}_1$ of Y contiaining y, $\Im_2 Cl f^{-1}(U)$ is not contained in a compact set of X.

A point $p \in Y$ is called a *pairwise singular* point if it is $\mathcal{L}_1$ singular point with respect to $\Im_2$ and $\mathcal{L}_2$ singular point with respect to $\Im_1$.

The set of all pairwise singular points of f is called the *pairwise singular set* of f and it is denoted by $S_B(f)$.

$\overline{S(f, \mathcal{L}_1, \Im_2)} = \{p \in Y \,/\, \forall\, U \in \mathcal{L}_1, \Im_2 Cl\, f^{-1}(U)$ is not compact$\}$

$\overline{S(f, \mathcal{L}_2, \Im_1)} = \{p \in Y \,/\, \forall\, U \in \mathcal{L}_1, \Im_1 Cl\, f^{-1}(U)$ is not compact$\}$.

$\overline{S_B(f)} = \overline{S(f, \mathcal{L}_1, \Im_2)} \cap \overline{S(f, \mathcal{L}_2, \Im_1)}$

Other definitions of singular sets as in case of bitopological spaces are the following:

2.2 Definition

$S'(f, \mathcal{L}_1, \Im_2) = \{p \in Y \,/\, \forall\, U \in \mathcal{L}_1, f^{-1}(\Im_2 Cl\ U)$ is not compact$\}$.

$S'(f, \mathcal{L}_2, \Im_1) = \{p \in Y \,/\, \forall\, U \in \mathcal{L}_2, f^{-1}(\Im_1 Cl\ U)$ is not compact$\}$.

$S_B'(f) = S'(f, \mathcal{L}_1, \Im_2) \cap S'(f, \mathcal{L}_2, \Im_1)$

2.3 Definition

$S(f, \mathcal{L}_1, \Im_2) = \{p \in Y \,/\, \forall\, U \in \mathcal{L}_1, \exists\, \mathcal{L}_1$ compact set F with $p \in F \subset U$ and $f^{-1}(F)$ is not $\Im_2$ compact$\}$.

$S(f, \mathcal{L}_2, \Im_1) = \{p \in Y \,/\, \forall\, U \in \mathcal{L}_2, \exists\, \mathcal{L}_2$ compact set F with $p \in F \subset U$ and $f^{-1}(F)$ is not $\Im_1$ compact$\}$.

$S_B(f) = S(f, \mathcal{L}_1, \Im_2) \cap S(f, \mathcal{L}_2, \Im_1)$.

and

2.4 Definition

$L(f, \mathcal{L}_1, \Im_2) = \cap\{\mathcal{L}_1 Cl\, f(X - F) / F \text{ is } \Im_2 \text{ compact}\}$.

$L(f, \mathcal{L}_2, \Im_1) = \cap\{\mathcal{L}_2 Cl\, f(X - F) / F \text{ is } \Im_1 \text{ compact}\}$.

$L_B(f) = L(f, \mathcal{L}_1, \Im_2) \cap \mathcal{L}(f, \mathcal{L}_2, \Im_1)$.

First we show that all the sets define above are equivalent.

2.5 Theorem

$S_B(f) = \mathcal{L}_B(f) = S_B'(f) = \overline{S_B(f)}$

Proof : If $p \notin L_B(f)$, then $p \notin L(f, \mathcal{L}_1, \Im_2) \cap L(f, \mathcal{L}_2, \Im_1)$.

Assume that $p \notin L(f, \mathcal{L}_1, \Im_2)$.

Then $p \notin \cap \{\mathcal{L}_1 Cl\, f(X - F) / F \text{ is } \Im_2 \text{ compact}\}$

Then $\exists\, U \in \mathcal{L}_1$ such that $p \in U$ and $U \cap f(X - F) = \phi$ for some $\Im_2-$ compact set F.

Or $f^{-1}(U \cap f(X - F)) = \phi$ for some $\Im_2-$compact set F;

Or $f^{-1}(U) \cap (X - F) = \phi$;

Or $f^{-1}(U) \subset F$.

For any $\mathcal{L}_1$ compact set K with $p \in K \subset U$, $f^{-1}(K) \subset F$ it follows that it is compact.

This implies that $p \notin S_B(f)$. Thus $S_B(f) \subseteq L_B(f)$

Now we show that $L_B(f) \subset S_B'(f)$.

Let $p \notin S'_B(f)$. Then $p \notin S'(f, \mathcal{L}_1, \Im_2) \cap S'(f, \mathcal{L}_2, \Im_1)$.

Assume that $p \notin S'(f, \mathcal{L}_1, \Im_2)$, then there exists $U \in \mathcal{L}_1$ with $p \in U$ and $f^{-1}(\Im_2 Cl\, U)$ is $\Im_2$ compact.

$f(X - f^{-1}(\overline{U})) = f(X) \cap (Y - \overline{U})$.

$p \in U$ and $U \cap [f(X) \cap (Y - U)] = \phi$ $p \notin L_B(f, \mathcal{L}_1, \Im_2)$.

Next, take $p \in S_B'(f)$. Then $p \in S'(f, \mathcal{L}_1, \Im_2) \cap S'(f, \mathcal{L}_2, \Im_1)$.

To show that $p \in \overline{S_B(f)}$, we show that $p \in \overline{S(f, \mathcal{L}_1, \Im_2)}$。

let $p \in U$, with $U \in \mathcal{L}_1$ choose V with $\mathcal{L}_2 Cl\, V \subset U$

$\Im_2 Cl\, f^{-1}(U) \supseteq f^{-1}(\mathcal{L}_2 Cl\, V)$.

If $\Im_2 Cl\, f^{-1}(U)$ is compact then $f^{-1}(\mathcal{L}_2 Cl\, V)$ is also compact.

$p \in \overline{S(f, \mathcal{L}_1, \Im_2)}$ Similarly it is shown that $p \in \overline{S(f, \mathcal{L}_1, \Im_1)}$.

Hence $p \in S_B(f)$ implies $S_B'(f) \subseteq S_B(f)$.

Finally we show that, $p \in S_B(f)$.

Let $p \in \overline{S_B(f)}$, take $U \in \mathcal{L}_1$ with $p \in U$. Then $\Im_2 Cl\, f^{-1}(U)$ is not compact.

Choose $V \subset U$ with $p \in V \in \mathcal{L}_2$ with $\mathcal{L}_2$ Cl $V \subset U$

Then $\Im_2 Cl\, f^{-1}(V) \subset f^{-1}(\mathcal{L}_2 Cl\, V)$

Note that $\Im_2 Cl\, f^{-1}(V)$ is not compact. This implies that $f^{-1}(\Im_2 Cl\, V)$ cannot be compact. This implies that $p \in V \subset \mathcal{L}_2$ Cl $V \subset U$ satisfying that $\mathcal{L}_2$ Cl V is compact but $f^{-1}(\mathcal{L}_2 Cl\, V)$ is not compact.

So $p \in S_B(f)$.

This implies that $\overline{S_B(f)} \subset S_B(f)$.

From above we conclude that

$S_B(f) = L_B(f) = S_B'(f) = S_B(f)$.

Throughout this paper, we use the following definition of a pairwise singular map. let f: $(X, \Im_1, \Im_2) \to (Y, \mathcal{L}_1, \mathcal{L}_2)$ be a pairwise continuous map where X, Y are pairwise locally compact, pairwise Haudorff spaces. Then a point $y \in Y$ is called $\mathcal{L}_1$ *singular point* with respect to $\Im_2$ if for each open set $U \in L_1$ of Y contiaining y, $\Im_2 Cl\, f^{-1}(U)$ is not in a compact set of X.

A point is *pairwise singular* point if it is L_1singular point with respect to $\Im_2$ and $\mathcal{L}_2$ singular point with respect to $\Im_1$.

i.e. $S_B(f) = S(f, \mathcal{L}_1, \Im_2) \cap S(f, \mathcal{L}_2, \Im_1)$

where, $S(f, \mathcal{L}_1, \Im_2) = \{p \in Y / \forall\, U \in \mathcal{L}_1, \Im_2 Cl f^{-1}(U)$ is not compact$\}$ and $S(f, \mathcal{L}_2, \Im_1) = \{p \in Y / \forall U \in \mathcal{L}_2, \Im_1 Cl\, f^{-1}(U)$ is not compact$\}$.

2.6 Example

Consider the sine map

$$\sin : (R, u, \Im_d) \to (R, u, \Im_d)$$

where u denotes the usual topology on R and $\Im_d$ denotes the discrete topology on R. The sine map is pairwise continuous and since it has non-compact fibres, it is a pairwise singular map.

3. Properties of Singular Maps

In the present section we discuss a few properties of

pairwise singular maps. The behavior of pairwise singular maps is seen in relation to the product , composition and restriction etc. an analogue of pasting lemma is also obtained for pairwise singular maps.

3.1 Theorem

Let $f : (X, \Im_1, \Im_2) \to (Y, \mathcal{L}_1, \mathcal{L}_2)$ and $g: (Y, \mathcal{L}_1, \mathcal{L}_2) \to (Z, \mathcal{L}_1', \mathcal{L}_2')$ be pairwise singular maps. Then the composition gof : $(X, \Im_1, \Im_2) \to (Z, \mathcal{L}_1', \mathcal{L}_2')$ is a pairwise singular map.

Proof: To show that gof is pairwise singular map, we prove the following cases :

Case I: $y \in Z$ is a $(\mathcal{L}_1', \Im_2)$ singular point of gof.

Case II: $y \in Z$ is a $(\mathcal{L}_2', \Im_1)$ singular point of gof.

Case I: First we show that gof is $(\mathcal{L}_1', \Im_2)$ singular. Take $p \in Z$ and $U' \in \mathcal{L}_1'$ with $p \in U'$.

Consider, $\Im_2\text{Cl}\,(g \circ f)^{-1}(U') = \Im_2\text{Cl}\,(f^{-1}(g^{-1}(U'))$.

Since $g : Y \to Z$ is pairwise continuous, $g^{-1}(U') \in \mathcal{L}_1$. Since f is pairwise singular, by definition of $(\mathcal{L}_1', \Im_2)$ singular map $\Im_2 f^{-1}(g^{-1}(U'))$ is not compact in X. This implies that $g \circ f : X \to Z$ is a $(\mathcal{L}_1', \Im_2)$ singular map. Similarly, it is proved that $g \circ f : X \to Z$ is $(\mathcal{L}_2', \Im_1)$ singular.

3.2 Theorm

Let $f : (X, \Im_1, \Im_2) \to (Y, \mathcal{L}_1, \mathcal{L}_2)$ and $h: (X', \Im_1', \Im_2') \to (Y', \mathcal{L}_1', \mathcal{L}_2')$ be pairwise continuous maps. Then the map $f \times h : (X \times X', \Im_1 \times \Im_1', \Im_2 \times \Im_2') \to (Y \times Y', \mathcal{L}_1 \times \mathcal{L}_1', \mathcal{L}_2 \times \mathcal{L}_2')$ defined by $(f \times h)(x, x') = (f(x), h(x'))$, where $(x, x') \in X \times X'$ is pairwise singular if and only if either f or h is pairwise singular.

Proof: let $f \times h : X \times X' \to Y \times Y'$ be pairwise singular. If f is not pairwise singular then there is a point $y \in Y$ which is not a pairwise singular point. Then there are three cases.

Case I: $y \in Y$ is not a $(\mathcal{L}_1, \Im_2)$ singular point of f.

Case II: $y \in Y$ is not a $(\mathcal{L}_2, \Im_1)$ singular point of f.

Case III: $y \in Y$ is neither $(\mathcal{L}_1, \Im_2)$ nor a $(\mathcal{L}_2, \Im_1)$ singular point of f.

Case I: Since $y \in Y$ is not a $(\mathcal{L}_1, \Im_2)$ singular point of f, there exists $U \in \mathcal{L}_1$ with $y \in U$ satisfying that $\Im_2\text{Cl}\,f^{-1}(U)$ is compact. Let $y' \in Y'$ and $U' \in L_1'$ with $y' \in U'$. Since $(y, y') \in Y \times Y'$ is a $(\mathcal{L}_1 \times \mathcal{L}_1', \Im_2 \times \Im_2')$ singular point of $f \times h$, we conclude that $(\Im_2 \times \Im_2')\,\text{Cl}\,(f \times h)^{-1}(U \times U')$ is not compact. Since $(\Im_2 \times \Im_2')\,\text{Cl}\,(f \times h)^{-1}(U \times U') = \Im_2\,\text{Cl}\,f^{-1}(U) \times \Im_2'\,\text{Cl}\,h^{-1}(U')$.

It follows that $\Im_2'\,\text{Cl}\,h^{-1}(U')$ is not compact. Thus h is a $(\mathcal{L}_1, \Im_2)$ singular map.

The proof of other cases follows similarly. This implies that h is a pairwise singular map. The proof of the converse is left to the reader.

3.3 Definition

Let $(X, \Im_1, \Im_2)$ be a bitopological space and $A \subseteq X$ then A is called a *pairwise L-Subspace* of X if the restriction of each pairwise singular map $f : (X, \Im_1, \Im_2) \to (Y, \mathcal{L}_1, \mathcal{L}_2)$ on $(A, \Im_1, \Im_2)$ is pairwise singular where $(Y, \mathcal{L}_1, \mathcal{L}_2)$ is any bitopological space.

3.4 Theorem

Let $(X, \Im_1, \Im_2)$ be any bitopological space. let K be $\Im_1, \Im_2$ compact set of X. Then $X - K$ is a pairwise L-subspace of X.

Proof: let $f : X \to Y$ be a pairwise singular map, where Y is pairwise locally compact space.

The restriction of $f : (X, \Im_1, \Im_2) \to (Y, \mathcal{L}_1, \mathcal{L}_2)$ is denoted by $g : (X - K, \Im_1, \Im_2) \to (Y, \mathcal{L}_1, \mathcal{L}_2)$. To prove that g is a pairwise singular map we show that

Case I: g is a $(\mathcal{L}_1, \Im_2)$ singular map.

Case II: g is a $(\mathcal{L}_2, \Im_1)$ singular map.

To show that g is a $(\mathcal{L}_1, \Im_2)$ singular map, take a non empty open set $U \in \mathcal{L}_1$

Case I: If $g^{-1}(U) = f^{-1}(U)$, then since f is pairwise singular map, $g^{-1}(U)$ is not contained in any $\Im_2$ compact set of $X - K$.

Case II : If $g^{-1}(U) \neq f^{-1}(U)$ then $f^{-1}(U) \subset g^{-1}(U) \cup K$.

If $g^{-1}(U)$ is not a $(\mathcal{L}_1, \Im_2)$ singular map then $g^{-1}(U)$ is contained in a $\Im_2$ compact set F of $X - K$.

Then $f^{-1}(U) \subset F \cup K$ which is a union of two $\Im_2$ compact sets of X. This implies that $f^{-1}(U)$ is contained in a $\Im_2$ compact set of X which is a

contradiction. Hence g is a ($\mathcal{L}_1$, $\Im_2$) singular map. The proof of case II is similar.

Since the pairwise singular map and the bitopological space Y are arbitrarry, X − K is a pairwise L-subspace of X.

3.5 Theorm

Let X be a bilopological space and A, B be two pairwise L-subspace of X with either A or B $\Im_1$, $\Im_2$ closed in X. Then A ∪ B is a pairwise L-subspace of X.

Proof: let f : (X, $\Im_1$, $\Im_2$) → (Y, $\mathcal{L}_1$, $\mathcal{L}_2$) be a pairwise singular map, where Y is a pairwise locally compact bitopological space. Assume that A is a $\Im_1$, $\Im_2$ closed L-subspace of X. f_A, f_B and $f_{A\cup B}$ denote respectively the restrictions of f on A, B and A ∪B. Since A and B are pairwise L - subspaces, f_A and f_B are pairwise singular maps. K be a $\Im_1$, $\Im_2$ compact set of A ∪B. To prove that $f_{A\cup B}$ is a pairwise singular map i.e. $f_{A\cup B}$: (A∪B, $\Im_1$, $\Im_2$) → (Y, $\mathcal{L}_1$, $\mathcal{L}_2$) is a ($\mathcal{L}_1$, $\Im_2$) and ($\mathcal{L}_2$, $\Im_i$) singular map, we proceed as below.

First we show that $f_{A\cup B}$ is ($\mathcal{L}_1$, $\Im_2$) singular map. Take U ∈ $\mathcal{L}_1$. Then $f^{-1}_{A\cup B}(U) = f^{-1}_A(U) \cup f^{-1}_B(U)$

If $f^{-1}_{A\cup B}(U)$ is contained in a compact set F of $\Im_2$ in A ∪ B then $f^{-1}_A(U)$ is contained in F ∩ A. Since F ∩ A is a closed subspace of F, it is compact. Since $f^{-1}_A(U) \subset F \cap A$, we have a contradiction to the assumption that f_A is a pairwise singular map. Similarly it is proved that $f_{A\cup B}$: (A ∪ B, $\Im_1$*, $\Im_2$*) → (Y, $\mathcal{L}_1$, $\mathcal{L}_2$) is a ($\mathcal{L}_2$, $\Im_i$) singular map. Since the map *f* and the bitopological space Y are arbitrary, A ∪ B is a pairwise L-subspace of X.

3.6 Theorm

Let X be a bitopological space and let A be a $\Im_1$, $\Im_2$ closed L-subspace of X. If B is a subset of X such that A ⊆ B, then B is also an pairwise L-subspace of X.

Proof: let f : X → Y be a pairwise singular map and f_A and f_B be restrictions of *f* on A and B respectively. Since A is pairwise L-subspace of X, f_A is a pairwise singular map.

To prove that f_B is a pairwise singular map, we show that f_B : (B, $\Im_1$, $\Im_2$) → (Y, $\mathcal{L}_1$, $\mathcal{L}_2$) is pairwise singular map i.e. it is a ($\mathcal{L}_1$, $\Im_2$) and ($\mathcal{L}_2$, $\Im_1$) singular map. First we prove that f_B is ($\mathcal{L}_1$, $\Im_2$) singular map. Take an open set U ∈ $\mathcal{L}_1$ in Y. Since A ⊆B,

$$f^{-1}_A(U) \subseteq f^{-1}_B(U).$$

If $f^{-1}_B(U)$ is contained in a compact set F of B then $f^{-1}_A(U)$ is contained in F ∩ A of $\Im_2$ which is a closed set of a compact space and hence compact. This contradicts the hypothesis that f_A is a pairwise singular map. Hence f_B is a ($\mathcal{L}_1$, $\Im_2$) singular map. Similarly it is a ($\mathcal{L}_2$, $\Im_1$) singular map.

Since the pairwise singular map *f* and the space Y are arbitrary, B is a pairwise L-subspace of X.

The following proposition gives an analogue of Pasting lemma for pairwise singular maps.

3.7 Theorm

Let X, Y be pairwise locally compact spaces and A, B be $\Im_1$, $\Im_2$ closed sets of X such that A∪B = X. let f : (A, $\Im_1$', $\Im_2$') → (Y, $\mathcal{L}_1$, $\mathcal{L}_2$) and g : (B, $\Im_1$'', $\Im_2$'') → (Y, $\mathcal{L}_1$, $\mathcal{L}_2$) be two pairwise singular maps such that $f(x) = g(x)$ for $x \in A\cap B$. Then the map h : (X, $\Im_1$, $\Im_2$) → (Y, $\mathcal{L}_1$, $\mathcal{L}_2$) defined by

$$h(x) = \begin{cases} f(x): if\ x \in A \\ g(x): if\ x \in B \end{cases}$$

is pairwise singular.

Proof : To prove that *h* is pairwise singular, we show that *h* is ($\mathcal{L}_1$, $\Im_2$) singular and ($\mathcal{L}_2$, $\Im_i$) singular. First we show that *h* is ($\mathcal{L}_1$, $\Im_2$) singular. let U be a $\mathcal{L}_1$ open set of Y.

Then $h^{-1}(U) = f^{-1}(U) \cup g^{-1}(U)$

or

$$\Im_2 Cl\, h^{-1}(U) = \Im_2' Cl\, f^{1}(U) \cup \Im_2'' Cl\, g^{-1}(U)$$

Since f and g are pairwise singular maps, and $\Im_2$'Cl $f^{-1}(U)$ and $\Im_2$''Cl$g^{-1}(U)$ both are non compact. Hence $\Im_2$Cl$h^{-1}(U)$ is also non compact. Since the open set U ∈ $\mathcal{L}_1$ in Y is arbitrary, h is a ($\mathcal{L}_1$, $\Im_2$) singular map. The proof of case (ii) i.e. h is ($\mathcal{L}_2$, $\Im_1$) singular map follows similarly. This implies that h is a pairwise singular map.

4. Pairwise Singular Maps And Various Structures

In this section the behaviour of pairwise singular maps is seen in relation to the G-spaces, Cone and twisted product etc.

4.1 Definition

A pairwise G-space consists of the following :

A Bitopological space $(X, \Im_1, \Im_2)$, a bitopological group $(G, \mathcal{L}_1, \mathcal{L}_2)$ and a pairwise continuous map $\theta: G \times X \rightarrow X$ satisfying

$\theta(e, x) = x$

$\theta(g_1, \theta(g_2, x)) = \theta(g_1 g_2, x)\ \forall\, x \in X, g_1, g_2 \in G.$

As in the case of a single topological space, the point $\theta(g, x)$ is denoted by gx. The pairwise orbit of a point $x \in X$ is denoted by $G_B(x) = \{gx / g \in G\}$ the collection X/G of pairwise orbits is given two topologies and thus becomes a bitopological space. It is called the *Bi-orbit space* of $x \in X$. The map $q : X \rightarrow X/G$ taking x to its orbit is called the *pairwise orbit map*.

4.2 Theorm

Let X, Y be pairwise G-spaces with G-$\Im_1$, $\Im_2$ compact bitopological group and $f : X \rightarrow Y$ be a pairwise singular equivariant map. Then the induced map $f_G : X/G \rightarrow Y/G$ is pairwise singular.

Proof: Consider the following commutative diagram :

$$\begin{array}{ccc} (X, \Im_1, \Im_2) & \xrightarrow{\quad f \quad} & (Y, \mathcal{L}_1, \mathcal{L}_2) \\ q_x\downarrow & & \downarrow q_y \\ (X/G, \Im_1', \Im_2') & \xrightarrow[\quad f_G \quad]{} & (Y/G, \mathcal{L}_1', \mathcal{L}_2') \end{array}$$

where q_x and q_y are pairwise orbit maps.

To show that f_G is a pairwise singular, we prove the following

Case I : $G(y) \in Y/G$ is a $(\mathcal{L}_1', \Im_2')$ singular point of f_G.

Case II : $G(y) \in Y/G$ is a $(\mathcal{L}_2', \Im_1')$ is a singular point of f_G.

Case I : Suppose $G(y) \in Y/G$ is not a $(\mathcal{L}_1', \Im_2')$ singular point of f_G. Then there exists an open set U of $\mathcal{L}_1'$ containing $G(y)$ such that $\Im_2'\mathrm{Cl}\, f_G^{-1}(U)$ is compact.

Since q_x is a $\Im_2'$-compact map, $q_x^{-1}(\Im_2'\mathrm{Cl}(f_G^{-1}(U)))$ is $\Im_2$-compact. From continuity of q_x and commutativity of the above diagram it follows that

$q_x^{-1}(\Im_2'\mathrm{Cl}(f_G^{-1}(U))) \supseteq \Im_2\mathrm{Cl}\, q_x^{-1}(f_G^{-1}(U)) = \Im_2\mathrm{Cl}(f^{1}(q_y^{-1}(U)))$

Thus $\Im_2\mathrm{Cl}(f^{1}(q_y^{-1}(U)))$ is $\Im_2$ compact which is a contradiction to the hypothesis that f is $(\mathcal{L}_1, \Im_2)$ singular.

The proof of case II follows similarly. This implies that f_G is a pairwise singular map.

let $(G, \Im_1, \Im_2)$ be a bitopological group. Assume that G is $\Im_1$, $\Im_2$ Hausdorff. let H be a $\Im_1$, $\Im_2$ compact subgroup of G and $(X, \mathcal{L}_1, \mathcal{L}_2)$ be a pairwise H-space. Then for $h \in H$ and $(g, x) \in G \times X$, $h(g, x) = (gh^{-1}, hx)$ defines an action of H on $G \times X$ which is pairwise continuous. The pairwise orbit space $(G \times H) / H$ is called the *pairwise twisted product* of G and X. For pairwise H-spaces X and Y, a pairwise equivariant map $f : (X, \mathcal{L}_1, \mathcal{L}_2) \rightarrow (Y, \mathcal{L}_1', \mathcal{L}_2')$ determines a pairwise continuous map $f_H : G\times_H X \rightarrow G\times_H Y$ which sends $[g, x] \in G\times_H X$ to $[g, f(x)]$ in $G\times_H Y$.

4.3 Corollary

Let $f : X \rightarrow Y$ be a pairwise singular map then $f_H : G \times_H X \rightarrow G\times_H Y$ is also a pairwise singular map.

Proof: Consider the following commutative diagram

$$\begin{array}{ccc} (G \times X, \Im_1, \Im_2) & \xrightarrow{\quad I_G \times f \quad} & (G \times Y, \mathcal{L}_1, \mathcal{L}_2) \\ q_x\downarrow & & \downarrow q_y \\ (G \times HX, \Im_1', \Im_2') & \xrightarrow[\quad f_H \quad]{} & (G \times HY, \mathcal{L}_1', \mathcal{L}_2') \end{array}$$

where q_x and q_y are pairwise orbit maps.

We have to show that f_H is a pairwise singular map. i.e.

Case I : f_H is $(\mathcal{L}_1', \Im_2')$ singular.

Case II : f_H is $(\mathcal{L}_2', \Im_1')$ singular.

Case I : Suppose f_H is not $(\mathcal{L}_1', \Im_2')$ singular. Then

there is an open set U ∈ $\mathcal{L}_1$' in G $\times_H$ Y such that $\Im_2$'Cl f_H^{-1}(U) is compact. Since q_x is a $\Im_2$' compact map, $q_x^{-1}(\Im_2'\mathrm{Cl}f_H^{-1}(U))$ is also compact. From continuity of q_x and commutativity of the above diagram, it follows that

$q_x^{-1}(\Im_2'\mathrm{Cl}(f_H^{-1}(U))) \supseteq \Im_2\mathrm{Cl}q_x^{-1}(f_H^{-1}(U)) = \Im_2\mathrm{Cl}(qy\ \mathrm{O}(IG \times f))^{-1}(U)$.

Since $\Im_2\mathrm{Cl}(q_y \circ (I_G \times f))^{-1}(U)$ is $\Im_2$ compact, a contradition to the hypothesis that $(I_G \times f)$ is pairwise singular is obtained. This implies that f_H is ($\mathcal{L}_1$', $\Im_2$') singular. The proof of case (II) follows similarly. Thus f_H is pairwise singular map.

4.4 Definition

Let (X, $\Im_1$, $\Im_2$) be a bitopological space. The pariwise cone (TX, $\mathcal{L}_1$, $\mathcal{L}_2$) is the quotient bitopological space X × I/A, where A = X × {1}.

4.5 Theorem

Let f : X → Y be a pairwise singular map where X and Y are pairwise locally compact spaces. If the quotient map p : X × I → TX is a $\Im_1$, $\Im_2$ compact map, then the induced map $T_f : T_X \to T_Y$ is a pairwise singular map.

Proof: Consider the following commutative diagram

$$\begin{array}{ccc} (X \times I, \Im_1, \Im_2) & \xrightarrow{f \times I} & (Y \times I, \mathcal{L}_1, \mathcal{L}_2) \\ p \downarrow & & \downarrow q \\ (TX, \Im_1', \Im_2') & \xrightarrow[T_f]{} & (TY, \mathcal{L}_1', \mathcal{L}_2') \end{array}$$

Now, we have to show that T_f is pairwise singular. i.e.

Case I : T_f is ($\mathcal{L}_1$', $\Im_2$') singular.

Case II : T_f is ($\mathcal{L}_2$', $\Im_1$') singular.

Case I : Suppose that T_f is not ($\mathcal{L}_1$', $\Im_2$') singular. Then there exists an open set U ∈ $\mathcal{L}_1$' of T_Y satisfying that $\Im_2'\mathrm{Cl}T_f^{-1}(U)$ is compact. Since p is a $\Im_2$' compact mapping $p^{-1}(\Im_2'\mathrm{Cl}T_f^{-1}(U))$ is $\Im_2$-compact. From continuity of p and commutativity of the above diagram, it follows that $p^{-1}(\Im_2'\mathrm{Cl}T_f^{-1}(U)) \supseteq \Im_2\mathrm{Cl}$ $(p^{-1}(T_f^{-1}(U))) = \Im_2\mathrm{Cl}\ [(f \times I)^{-1}(q^{-1}(U))]$ Thus $\Im_2\mathrm{Cl}\ (f \times I)^{-1}q^{-1}(U)$ is compact, a contradiction to the hypothesis that f is a pairwise singular map. This implies that T_f is a ($\mathcal{L}_1$', $\Im_2$') singular. The proof of case (II) is similar. So T_f is a pairwise singular map.

5. Discussion and Conclusion

We discussed properties of pairwise singular maps. The behavior of pairwise singular maps in relation to the product , composition and restriction etc. The set of all pairwise singular maps from R to R which are periodic in addition is closed under the addition,product ,division and composition of maps.

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The Convergence of 1-Periodic Branched Continued Fraction of the Special Form in Parabolic Regions

Dmytro I. Bodnar[1] and Mariia M. Bubniak[2]

1. st. Samchuk 32/6 Ternopil 46002 Ukraine

2. st. Kyivska 12/155 Ternopil 46023 Ukraine

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Abstract: Branched continued fractions are one of the multidimensional generalization of the continued fractions. Branched continued fractions with not equivalent variables are an analog of the regular C-fractions for multiple power series. We consider 1-periodic branched continued fraction of the special form which is an analog fraction with not equivalent variables if the values of that variables are fixed. We establish an analog of the parabola theorem for that fraction and estimate truncation error bounds for that fractions at some restrictions. We also propose to use weight coefficients for obtaining different parabolic regions for the same fraction without any additional restriction for first element.

Key words: Continued fractions, 1-periodic branched continued fraction of special form, convergence, uniform convergence, truncation error bounds.

1. Introduction

Establishing the convergence regions is one of the investigation methods of convergence for continued fraction

$$1+\mathop{D}_{n=1}^{\infty}\frac{a_n}{1}=1+\cfrac{a_1}{1+\cfrac{a_2}{1+\ddots}}, \qquad (1)$$

where $a_n \in C$. The sequence of nonempty sets $\{\Omega_n\}_{n=1}^{\infty}$, $\Omega_n \subseteq C$ are called the sequence of convergence regions for fraction (1), if conditions: $a_n \in \Omega_n$ for $n \geq 1$ guarantee the convergence of that fraction. If $\Omega_n = \Omega$ ($n \geq 1$), then the region Ω called the simple convergence region of continued fraction (1). It is often considered the circular, angular, parabolic regions in the analytic theory of continued fraction.

In particular, the research review concerning the parabolic regions is given in the next monographs [10, 13, 14, 15].

We consider multidimensional generalization of parabolic theorem for 1-periodic branched continued fraction (BCF) of the special form [10, p. 99].

Analogs of parabolic theorems are established for different forms of branched continued fraction (multidimensional generalization of continued fraction). D.I. Bodnar [4] proved multidimensional analogs of parabolic theorems for BCF of general form with N branches

$$1+\mathop{D}_{k=1}^{\infty}\sum_{i_k=1}^{N}\frac{a_{i(k)}}{1}=1+\sum_{i_1=1}^{N}\cfrac{a_{i(1)}}{1+\sum\limits_{i_2=1}^{N}\cfrac{a_{i(2)}}{1+\ddots}}, \qquad (2)$$

where $a_{i(k)} \in C$, $i(k)=i_1 i_2 \ldots i_k$ – multi index ($1 \leq i_k \leq N$, $k \geq 1$). T.M. Antonova [1] established multidimensional generalization of parabolic theorem with the following condition: if elements from different "floors" belong to different parabolic regions. R.I. Dmytryshyn [9] proved the convergence of multidimensional generalization g-fractions in the union of parabolic regions and established the

Corresponding author: Mariia M. Bubniak, Ph.D. student, research field: 1-periodic branched continued fractions of the special form. E-mail: maria.bubnyak@gmail.com.

truncation error bounds for that fractions in some bounded part in the parabolic region. O.Ye. Baran [3] investigated the convergence of branched continued fraction of the special form

$$b_0 + \mathop{D}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}}{b_{i(k)}} = b_0 + \sum_{i_1=1}^{i_0} \frac{a_{i(1)}}{b_{i(1)} + \sum_{i_2=1}^{i_1} \frac{a_{i(2)}}{b_{i(2)} + \ddots}}, \quad (3)$$

where $a_{i(k)}$, $b_{i(k)}$ – complex numbers, $i(k)$ – multi index, $1 \le i_k \le i_{k-1}$, $k \ge 1$, $i_0 = N$ – integer, if partial numerators are in corresponding parabolic regions and partial denominators – the half-plane. Kh.Yo. Kuchmins'ka [11] proved analogs of parabolic regions for two-dimensional continued fractions with complex elements

$$\mathop{D}_{k=1}^{\infty} \frac{a_{i,i}}{\Phi_i}, \quad \Phi_i = 1 + \mathop{D}_{j=1}^{\infty} \frac{a_{j+i,j}}{1} + \mathop{D}_{j=1}^{\infty} \frac{a_{i,i+j}}{1}.$$

2. Main Definitions and Methods

The object of our researching is 1-periodic branched continued fraction of special form, that we receive from (3), if we put: $a_{i(k)} = c_{i_k}$, $b_{i(k)} = 1$ ($1 \le i_k \le i_{k-1}$, $k \ge 1$), that is BCF next form

$$\left(1 + \mathop{D}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{c_{i_k}}{1}\right)^{-1} = \left(1 + \sum_{i_1=1}^{i_0} \frac{c_{i_1}}{1 + \sum_{i_2=1}^{i_1} \frac{c_{i_2}}{1 + \ddots}}\right)^{-1}, \quad (4)$$

where c_j – complex numbers for $j = \overline{1, N}$, $i_0 = N$ – integer. The n-th approximant of 1-periodic BCF (4) is

$$F_n = \left(1 + \mathop{D}_{i_k=1}^{n} \sum_{i_k=1}^{i_{k-1}} \frac{c_{i_k}}{1}\right)^{-1} \quad (n \ge 1;\ F_0 = 1).$$

We define

$$R_n^{(q)} = 1 + \mathop{D}_{k=1}^{n} \sum_{j_k=1}^{j_{k-1}} \frac{c_{j_k}}{1} = 1 + \sum_{j_1=1}^{j_0} \frac{c_{j_1}}{1 + \sum_{j_2=1}^{j_1} \frac{c_{j_2}}{1 + \ddots \sum_{j_n=1}^{j_{n-1}} \frac{c_{j_n}}{1}}}$$

as n-th tail q-th order of 1-periodic BCF (4) ($q = \overline{1, N}$; $n \ge 1$; $j_0 = q$; $R_0^{(q)} = 1$; $R_n^{(0)} = 1$). Obviously, that the tails $R_n^{(q)}$ ($n \ge 1$; $q = \overline{2, N}$) satisfy following recurrence expression

$$R_n^{(q)} = R_n^{(1)} + \sum_{s=2}^{q} \frac{c_s}{R_{n-1}^{(s)}} \quad (5)$$

We use

- technique of value set for establishing the value region of fraction (4) [12, p. 110];
- multidimensional analog of Stiltjes-Vitali Theorem from [4, theorem 2.17, p. 66] for proving uniform convergence of 1-periodic BCF;
- method of fundamental inequalities for establishing truncation error bounds [15].

3. Main Results

3.1 Useful Lemma

Let assume that the vector $\vec{v} = (v_1, v_2, \ldots, v_N)$ is given, where $v_j > 0$ – weight coefficients ($j = \overline{1, N}$) and $\sum_{j=1}^{N} v_j < 1$. We denote $\vartheta_j = 1 - \sum_{s=1}^{j} v_s$ .Then following lemma is valid.

Lemma 1. Let elements c_j ($j = \overline{1, N}$) of fraction (4) satisfy: $c_j \in P_j(\gamma)$ for $\gamma \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$, where

$$P_j(\gamma) = \left\{ z \in C : |z| - \Re\left(z e^{-2i\gamma}\right) \le 2 p_j \cos^2 \gamma \right\} \quad (6)$$

and $p_j = v_j \vartheta_j$. Then tails $R_n^{(j)}$ of 1-periodic BCF ($n \ge 0$, $j = \overline{1, N}$) belong to the half-planes $H(\vartheta_j; \gamma)$, where

$$H(\vartheta_j; \gamma) = \left\{ z \in C : \Re\left(z e^{-i\gamma}\right) \ge \vartheta_j \cos \gamma \right\}. \quad (7)$$

Proof. We consider the half-planes $1 + H(-v_s; \gamma) = H(\vartheta_j; \gamma)$ for any $j = \overline{1, N}$. Since $\sum_{j=1}^{N} v_j < 1$ next conditions are valid: $0 \notin H(\vartheta_j; \gamma)$ The set $c_j / H(\vartheta_j; \gamma)$ belongs to the circle

$$\Omega_j = \left\{ z \in C : \left| z - \frac{c_j e^{-i\gamma}}{2\vartheta_j \cos \gamma} \right| \le \frac{|c_j|}{2\vartheta_j \cos \gamma} \right\}.$$

Including $\Omega_j(\gamma) \subset H(-v_j; \gamma)$ is holding, if inequalities $\frac{|c_j|}{2\vartheta_j \cos \gamma} \le v_j \cos \gamma + \Re\left(\frac{c_j e^{-2i\gamma}}{2\vartheta_j \cos \gamma}\right)$ are valid.

They are equivalent to $c_j \in P_j(\gamma)$ ($j=\overline{1,N}$). Q.E.D.

3.2 Main Theorem and Consequence

We denote $G=\left\{z\in C:\left|\arg\left(z+\frac{1}{4}\right)\right|<\pi\right\}$.

Theorem. Let elements c_j of fraction (4) belong to parabolic regions: $P_j(\gamma)$ ($j=\overline{1,N}$), which are defined by formula (6). Then

(1) 1-periodic BCF (4) converges uniformly on any compact of the set P, where $P=P_1(\gamma)\times\ldots\times P_N(\gamma)$;

(2) the value region is the circle

$$K_N(\gamma)=\left\{z\in C:\left|z-\frac{e^{-i\gamma}}{2\vartheta_N\cos\gamma}\right|\le\frac{1}{2\vartheta_N\cos\gamma}\right\};$$

(3) if $\left|c_j\right|<\vartheta_j{}^2\cos^2\gamma$ for $j=\overline{2,N}$, then the truncation error bounds of (4) hold

$$\left|F_{n+m}-F_n\right|\le L\binom{N-1}{n+N-1}p^{n+1}\quad(n>0\,;\,m>0),$$

where $L=\frac{\prod_{j=1}^{N}M_j}{\vartheta_j{}^2\cos^2\gamma}$, $p=\max\limits_{j=\overline{1,N}}\{p_j\}$, $p_1=\left|\frac{1-\sqrt{1+4c_1}}{1+\sqrt{1+4c_1}}\right|$,

$p_j=\frac{|c_j|}{\vartheta_j{}^2\cos^2\gamma}$, $M_1=\frac{2\left|1+\sqrt{1+4c_1}\right|}{1-p_1}$, $M_j=\max\left\{1;\frac{|c_j|}{p_j\vartheta_j\cos\gamma}\right\}$.

Proof. We use multidimensional analog of Stiltjes-Vitali Theorem for proving uniform convergence of 1-periodic BCF. We are going to investigate the functional fraction following form

$$\left(1+\mathop{D}_{i_k=1}^{\infty}\sum_{i_k=1}^{i_{k-1}}\frac{z_{i_k}}{1}\right)^{-1}$$

and it's respective the sequence n-th approximants $\{F_n(z)\}_{n=1}^{\infty}$ and $z=(z_1,z_2,\ldots,z_N)$.

According to the previous lemma it holds $R_n^{(N)}(z)\subset H(\vartheta_N;\gamma)$. Since $F_n(z)=\left(R_n^{(N)}(z)\right)^{-1}$, then $F_n(z)\in K_N(\gamma)$, that is why the sequence of approximants $F_n(z)$ is bounded uniformly. We define $R=\min\limits_{j=\overline{1,N}}\left\{\frac{1}{4N};v_j\vartheta_j\right\}$ and the compact D, where $D=\left\{z\in C^N:\left|z_j\right|\le R\right\}$ on the set P. Obviously, that $D\subset P$. The functional fraction converges on the compact D by the multidimensional analog Worpitzky's Theorem [2]. It proves the uniform convergence of 1-periodic BCF (4) on any compact of the set P.

We use the following inequality for the truncation error bounds of (4)

$$\left|F_{n+m}-F_n\right|\le$$

$$\frac{1}{\left|R_{n+m}^{(N)}\right|\left|R_n^{(N)}\right|}\cdot\sum_{\substack{k_1+\cdots+k_N=n+1\\k_j\ge0(j=\overline{1,N})}}\frac{\left|c_1\right|^{k_1}\left|c_2\right|^{k_2}\cdots\left|c_N\right|^{k_N}}{\prod\limits_{j=1}^{N}\prod\limits_{r=1}^{k_j}\left(\left|R_{m+s_j-r}^{(j)}\right|\cdot\left|\hat{R}_{s_j-r}^{(j)}\right|\right)},$$

where $s_j=n-\sum_{l=j+1}^{N}k_l$; $\hat{R}_n^{(q)}=\begin{cases}R_n^{(q)}, & if\ n\ge0\\ 1, & if\ n=-1\end{cases}$,

that leads from difference between F_{n+m} and F_n ($n>0$, $m>0$). It was proved in [7].

Since tails $R_n^{(1)}$ is n-th approximant of 1-periodic continued fraction, $c_1\in P_1(\gamma)$ and $P_1(\gamma)\subset G$, we obtain $R_n^{(1)}=\frac{x^{n+2}-y^{n+2}}{x^{n+1}-y^{n+1}}$ for $n\ge0$, where $x=\frac{1}{2}\left(1+\sqrt{1+4c_1}\right)$, $y=\frac{1}{2}\left(1-\sqrt{1+4c_1}\right)$ – the attracting and the repelling fixed points of the linear fraction transformation $s(\omega)=1+c_1/\omega$. We use for proving the same scheme as in problem 13 [13, p. 49]. Otherwise, it holds $c_1=x\cdot y$. Since

$$\prod_{r=1}^{k_1}\left|R_{q-r}^{(1)}\right|\le\left|x\right|^{k_1}/(1-|y/x|)$$

for $q=s_1+m$ or $q=s_1$ thus we obtain for $1\le k_1\le n+1$

$$\frac{\left|c_1\right|^{k_1}}{\prod\limits_{r=1}^{k_1}\left(\left|R_{s_1+m-r}^{(1)}\right|\cdot\left|\hat{R}_{s_1-r}^{(1)}\right|\right)}\le M_1\left(\frac{\left|c_1\right|}{\left|x\right|^2}\right)^{k_1}\le M_1\left(\frac{\left|y\right|}{\left|x\right|}\right)^{k_1}\le M_1p_1^{k_1}$$

where

$$M_1=\frac{2\left|1+\sqrt{1+4c_1}\right|}{1-p_1},\quad p_1=\left|\frac{1-\sqrt{1+4c_1}}{1+\sqrt{1+4c_1}}\right|,\quad\sqrt{1}=1.$$

According to the lemma the next estimations hold for modulus of the tails: $\left|R_n^{(j)}\right|\ge\Re\left(R_n^{(j)}e^{-i\gamma}\right)\ge\vartheta_j\cos\gamma$.

Using the inequalities: $\frac{|c_j|}{\left|R^{(j)}_{s_j+m-r}\right|\cdot\left|\hat{R}^{(j)}_{s_j-r}\right|} \le \frac{|c_j|}{\vartheta_j^2 \cos^2\gamma}$ we obtain

$$\prod_{r=1}^{k+1} \frac{|c_j|}{\left|R^{(j)}_{s_j+m-r}\right|\cdot\left|\hat{R}^{(j)}_{s_j-r}\right|} \le M_j p_j^{k+1} \quad (1 \le k \le n+1),$$

where

$$M_j = \max\left\{1; \frac{|c_j|}{p_j \vartheta_j \cos\gamma}\right\}, \quad p_j = \frac{|c_j|}{\vartheta_j^2 \cos^2\gamma}.$$

We denote $p = \max_{j=1,N}\{p_j\}$ and obtain the final estimations of (4)

$$|F_{n+m} - F_n| \le \frac{1}{\left|R^{(N)}_{n+m}\right|\cdot\left|R^{(N)}_n\right|} \times$$

$$\sum_{k_1+\dots+k_N=n+1} M_1 p_1^{k_1} \dots M_N p_N^{k_N} \le L\begin{pmatrix} N-1 \\ n+N-1 \end{pmatrix} p^{n+1}$$

where $L = \prod_{j=1}^{N} M_j / \left(\left|R^{(N)}_{n+m}\right|\cdot\left|R^{(N)}_n\right|\right)$. Q.E.D.

This theorem generalizes the theorem 2 [6] if we define $\vec{v}$ in such way $\vec{v} = \left(\frac{1}{2N}, \frac{1}{2N}, \dots, \frac{1}{2N}\right)$.

If element $c_1 \in G$ and it is fixed this theorem can be improved.

Consequence. Let $N>1$ and elements c_j of the fraction (4) satisfy the conditions: $c_1 \in G$, $c_j \in P_j(\alpha_1)$, where

$$P_j(\alpha_1) = \left\{z \in C : |z| - \Re\left(ze^{-2i\alpha_1}\right) \le 2p_j \cos^2\alpha_1\right\}$$

where $p_j = \frac{1-\varepsilon}{2(N-1)}\left(\frac{1}{2} - \frac{(1-\varepsilon)(j-1)}{2(N-1)}\right)$ and the angle α_1 is defined as $2\alpha_1 = \begin{cases} \arg c_1, & if \ \arg c_1 \ne \pi \\ 0, & if \ \arg c_1 = \pi \end{cases}$, ε – some parameter ($0<\varepsilon<1$). Then

(1) 1-periodic BCF converges uniformly for c_2, ..., c_N if element c_1 is fixed;

(2) the value region is the circle

$$K = \left\{z \in C : \left|z - \frac{e^{-i\alpha_1}}{\varepsilon\cos\alpha_1}\right| \le \frac{1}{\varepsilon\cos\alpha_1}\right\}.$$

Proof. We take $\vec{v} = \left(\frac{1}{2}, \frac{1-\varepsilon}{2(N-1)}, \dots, \frac{1-\varepsilon}{2(N-1)}\right)$. Since $c_1 \in G$ and $G = \bigcup_{\gamma\in\left(-\frac{\pi}{2};\frac{\pi}{2}\right)} P_1(\gamma)$ for $p_1=1/4$, then $\exists\alpha_1 : \ c_1 \in P_1(\alpha_1)$. We choose and fix the element c_1. Since the main theorem with the conditions: $c_j \in P_j(\alpha_1)$ for $j = \overline{1, N}$ and obtain, that fraction (4) converges uniformly on any compact of the set P. We define $R = \max_{j=1,N}\left\{|c_j|\right\}$ and the compact D of the set P

$$D = \left\{z = (z_1, \dots, z_N) \in C^N : z_j \in P_j(\alpha_1) \cap \left\{|z_j| \le R\right\}\right\}$$

The fraction (4) converges on the compact D. The uniform convergence of BCF (5) according c_2, ..., c_N leads from the multidimensional generalization of Stiltjes-Vitali Theorem.

Taking into consideration, that $\vartheta_N = 1 - \sum_{s=2}^{N} v_j = \varepsilon/2$, we obtain the value region is $K(\varepsilon,\alpha_1)$. It is following from the main theorem. Q.E.D.

It was proved in [5] that $F = \left(\prod_{j=1}^{N} x_j\right)^{-1}$, where x_j – the attracting fixed point of the linear fractional transformation of $s_j = 1 + \frac{c_j / \prod_{s=1}^{j-1} x_s^2}{\omega}$ ($j = \overline{2, N}$) and $x_1 = \frac{1}{2}\left(1 + \sqrt{1+4c_1}\right)$ – the attracting fixed point respective of $s_1 = 1 + c_1/\omega$. It is equivalent $F = 1/u_N$, where u_N – the attracting fixed point of $\tau_N(\omega) = u_{N-1} + c_N/\omega$ and $u_1=x_1$.

3.3 Examples

Example 1. Let $N=2$, then we consider 1-periodic branched continued fractions of the following form

$$\left(1 + \cfrac{c_1}{1 + \cfrac{c_1}{1 + \cfrac{c_1}{1 + \ddots}}} + \cfrac{c_2}{1 + \cfrac{c_1}{1 + \cfrac{c_1}{1+\ddots}} + \cfrac{c_2}{1 + \cfrac{c_1}{1+\ddots} + \cfrac{c_2}{1+\ddots}}}\right)^{-1},$$

where $c_1=1+i$ and $c_2=i/8$. The convergence of this fraction leads from the main theorem for $\gamma=\pi/4$ and $\vec{v} = (1/2; 1/4)$. The calculation is in the next table

n	$R_n^{(1)}$	$R_n^{(2)}$	F_n
0	1	1	1
…			
14	1.69388+0.41847*i*	1.71306+0.48626*i*	0.54022-0.15335*i*
…			
19	1.69389+0.41849*i*	1.71303+0.48632*i*	0.54021-0.15336*i*

We are going to calculate the attracting fixed points: u_1, u_2 and the value F. It was proved in [5], that $\lim_{n\to\infty} R_n^{(1)} = u_1$, where $u_1 = \frac{1}{2}\left(1+\sqrt{1+4c_1}\right)$. Otherwise, u_1 is the solution of the equation $u_1^2 - u_1 - c_1 = 0$. The attractive fixed point of linear-fractional transformation $\tau_2(\omega) = u_1 + c_2/\omega$ is equal $u_2 = \frac{1}{2}\left(u_1 + \sqrt{u_1^2 + 4c_2}\right)$. That is why the value of BCF with 2 branches equals

$$F = (u_2)^{-1}$$
$$= \left(\frac{1}{4} + \frac{1}{4}\sqrt{1+4c_1} + \frac{1}{2}\sqrt{\frac{1}{2} + c_1 + 4c_2 + \frac{1}{2}\sqrt{1+4c_1}}\right)^{-1}.$$

And now we obtain: $u_1=1.6938+0.4187i$ for $n=14$ and $u_2=1.71303+0.4863i$ for $n=19$. Towards the end, the value of that fraction is equal $F=0.5402-0.15336i$.

Example 2. Let $c_1=i/6$ and $c_2=10+11i$. The convergence of that fraction leads from the main theorem for $\gamma=\pi/4$ and $\vec{v} = (1/2;1/4)$.

Analogically as in the previous example we obtain $u_1=1.0246+0.1588i$ for $n=6$ and $u_2=4.0728+1.6355i$ for $n=58$. The value of the fraction equals $F=0.2114-0.0849i$.

4. Conclusions

The uniform convergence of the 1-periodic branched continued fraction of the special form is proved in the parabolic regions. Truncation error bounds are established at some additional restrictions of the elements beginning from the second. We obtained the value set and the value of the 1-periodic branched continued fraction of the special form.

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A Comparison Results of Some Analytical Solutions of Model in Double Phase Flow through Porous Media

Olaniyi Samuel Iyiola, Gbenga Olayinka Ojo, Johnson Daddy Audu

Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, KFUPM, Dhahran, Saudi Arabia

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Abstract: In this paper, we consider Variational Iteration Method (VIM) and q-Homotopy Analysis Method (q-HAM) to solve the partial differential equation resulted from Fingero Imbibition phenomena in double phase flow through porous media. We further compare the results obtained here with the solution obtained in [12] using Adomian Decomposition Method. Numerical results are obtained, using Mathematica 9, to show the effectiveness of these methods to our choice of problem especially for suitable values of h and n.

Keyword: Variational iteration method, q-homotopy analysis method, double phase flow, porous media.

1. Introduction

We consider the following nonlinear partial differential equation resulting from Fingero-Imbibition phenomena in double phase flow through porous media [12]

$$\frac{\partial S}{\partial t} = \left(\frac{\partial S}{\partial x}\right)^2 + S\frac{\partial^2 S}{\partial x^2} \tag{1}$$

with appropriate initial condition.

Many years ago, various methods have been put to use successfully to obtain analytical solutions of both ordinary differential equations and partial differential equations such methods as Adomian Decomposition Method (ADM) [12], Variational IterationMethod (VIM) [1, 8, 20], Homotopy Perturbation Method (HPM) [7], and EXP-function Method [19]. One of the effective analytical approach to solving nonlinear differential equations is Homotopy Analysis Method (HAM) [13, 14]. However, the region of h of convergence is still questionable as to how large this region could be for fast convergence.

This paper therefore considers a more general form of HAM called q-HAM [6,17] and VIM, for effective comparison, to solve equation (1). Recently, q-HAM was succefully applied to obtain an aproximate solution of Riccati equation in [6] and it was proven that the convergence region of series solutions obtained by q-HAM is increasing as q is decreasing see also [17, 18]. Analytical solution of equation 1 is obtained using VIM and q-HAM subject to some appropriate initial condition. We further compare these results with the result obtained using ADM in [12] to affirm the reliability of these methods including numerical values. We obtain numerical results at the end with different values of n and h (n is the new parameter embedded in q-HAM) given the effect of both parameters on solution of fingero-imbibition phenomena in double phase flow through porous media.

2. The Idea of Variational Iteration Method

We present the basic idea of the Variational Iterative Method (VIM) in this section by considering the following nonlinear partial differential equation

$$\frac{\partial^m u(x,t)}{\partial t^m} + R[u(x,t)] + N[u(x,t)] = f(x,t) \tag{2}$$

where R is a linear operator, N is a nonlinear

Corresponding author: Olaniyi Samuel Iyiola, Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals. E-mail: samuel@kfupm.edu.sa.

operator and $f(x,t)$ is a given continuous function.

The idea is now to construct the following correction function for (2)

$$u_{n+1} = u_n +$$

$$+\int_{t_0}^{t} \lambda(t,\tau)\left[\frac{\partial^m u(x,t)}{\partial t^m} + R[u(x,t)] + N[u(x,t)] - f(x,t)\right] d\tau \tag{3}$$

where $\lambda = \lambda(t,\tau)$ is called the Lagrange multiplier which can be identified optimally by variational calculus and u_n is the nth term approximate solution.

The Lagrange multiplier λ in (3) of equation (2) ia calculated using

$$\lambda = \frac{(-1)^m (\tau - t)^{m-1}}{(m-1)!}. \tag{4}$$

See [9, 10] for the interpretation and determination of various Lagrange multipliers.

3. The Idea of q-Homotopy Analysis Method

Considering the following differential equation of the form

$$N\left[u(x,t)\right] - f(x,t) = 0 \tag{5}$$

where N is a nonlinear operator, (x,t) are independent variables, $f(x,t)$ is a known function and $u(x,t)$ is an unknown function.

$$(1-nq)L\left(\Phi(x,t;q) - u_0(x,t)\right)$$

$$= qhH(x,t)\left(N[\Phi(x,t;q)] - f(x,t)\right), \tag{6}$$

where $n \geq 1$, $q \in \left[0, \frac{1}{n}\right]$ denotes the so-called embedded parameter, L is an auxiliary linear operator, $h \neq 0$ is an auxiliary parameter, $H(x,t)$ is a non-zero auxiliary function.

Clearly, when $q = 0$ and $q = \frac{1}{n}$, equation (6) becomes

$$\Phi(x,t;0) = u_0(x,t) \quad and \quad \Phi(x,t;\frac{1}{n}) = u(x,t) \tag{7}$$

respectively.

So, as q increases from 0 to $\frac{1}{n}$, the solution $\Phi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$.

If $u_0(x,t)$, L, h, $H(x,t)$ are chosen appropriately, solution $\Phi(x,t;q)$ of equation(6) exists for $q \in \left[0, \frac{1}{n}\right]$.

Taylor series expansion of $\Phi(x,t;q)$ gives

$$\Phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m. \tag{8}$$

where

$$u_m(x,t) = \frac{1}{m!} \frac{\partial^m \Phi(x,t;q)}{\partial q^m}\bigg|_{q=0}. \tag{9}$$

We suppose that the auxiliary linear operator L, the initial guess u_0, the auxiliary parameter h and $H(x,t)$ are properly chosen such that the series 8 converges at $q = \frac{1}{n}$, then we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)\left(\frac{1}{n}\right)^m. \tag{10}$$

Define

$$\vec{u}_n = \left\{u_0(x,t), u_1(x,t), \cdots, u_n(x,t)\right\}. \tag{11}$$

Differentiating equation (6) m-times with respect to the (embedding) parameter q, then evaluating at $q = 0$ and finally dividing them by $m!$, we have the so called mth-order deformation equation see Lioa [13, 14, 16] as

$$L\left[u_m(x,t) - \chi_m^* u_{m-1}(x,t)\right] = hH(x,t)\breve{R}_m\left(\vec{u}_{m-1}\right). \tag{12}$$

with initial conditions

$$u_m^{(k)}(x,0) = 0, \qquad k = 0,1,2,\ldots,m-1. \tag{13}$$

where

$$\breve{R}_m\left(\vec{u}_{m-1}\right) = \frac{1}{(m-1)!} \frac{\partial^{m-1}\left(N[\Phi(x,t;q)] - f(x,t)\right)}{\partial q^{m-1}}\Big|_{q=0} \tag{14}$$

and

$$\chi_m^* = \begin{cases} 0 & m \leq 1 \\ n & otherwise \end{cases} \tag{15}$$

Remark 1. It should be emphasized that $u_m(x,t)$

for $m \geq 1$, is governed by the linear operator (12) with the linear boundary conditions that come from the original problem. The existence of the factor $\left(\frac{1}{n}\right)^m$ gives more chances for better convergence, faster that the solution obtained by the standard HAM. Ofcourse, when $n = 1$, we are in the case of the standard HAM.

4. Applications

4.1 Application of VIM to Fingero-Imbibition phenomena

We consider equation (1) with the initial condition

$$S(x,0) = 1 - \xi e^{-x}$$

where $\xi = 1.11$. This problem was solved in [12] using ADM.

In the case of VIM, we use $S_0(x,t) = S(x,0) = 1 - \xi e^{-x}$ as the initial approximation and

$$m = 1, \quad R[S(x,t)] = 0,$$

$$N[S(x,t)] = -\left(\frac{\partial S}{\partial x}\right)^2 - S\frac{\partial^2 S}{\partial x^2} \quad and \quad f(x,t) = 0.$$

Hence, using (3), we have the correction function as

$$u_{n+1} = u_n + \int_0^t \lambda(t,\tau)\left[\frac{\partial S(x,t)}{\partial \tau} - \left(\frac{\partial S}{\partial x}\right)^2 - S\frac{\partial^2 S}{\partial x^2}\right]d\tau \tag{16}$$

By (4), we obtain $\lambda = -1$. Therefore, we get the followings from the iteration formular given by (16)

$$S_1(x,t) = 1 - \xi e^{-x} + 2\xi^2 e^{-2x}t - \xi e^{-x}t \tag{17}$$

$$S_2(x,t) =$$

$$1 - \xi e^{-x} + 2\xi^2 e^{-2x}t - \xi e^{-x}t + 6\xi^2 e^{-2x}t^2 - \frac{1}{2}\xi e^{-x}t^2$$
$$-9\xi^3 e^{-3x}t^2 - 6\xi^3 e^{-3x}t^3 + \frac{2}{3}\xi^2 e^{-2x}t^3 + \frac{32}{2}\xi^4 e^{-4x}t^4. \tag{18}$$

Similarly, $S_m(x,t)$ for $m = 3,4,5,\cdots$ can be obtained using Mathematica 9.

4.2 Application of q-HAM to Fingero-Imbibition phenomena

Using $S_0(x,t) = 1 - \xi e^{-x}$ as initial approximation. We have the solution to problem (1) given as see [] for details

$$S_1(x,t) = h\xi e^{-x}t - 2h\xi^2 e^{-2x}t \tag{19}$$

$$S_2(x,t) =$$
$$h(n+h)\xi e^{-x}t - 2h(n+h)\xi^2 e^{-2x}t - \frac{h^2\xi e^{-x}t^2}{2}$$
$$+6h^2\xi^2 e^{-2x}t^2 - 9h^2\xi^3 e^{-3x}t^2 \tag{20}$$

$$S_3(x,t) = h(n+h)^2\xi e^{-x}t -$$
$$-2h(n^2 - h^2)\xi^2 e^{-2x}t - h^2(n+h)\xi e^{-x}t^2$$
$$+12h^2(n+h)\xi^2 e^{-2x}t^2 -$$
$$-18h^2(n+h)\xi^3 e^{-3x}t^2 + \frac{h^3\xi e^{-x}t^3}{6}$$
$$-\frac{28h^3\xi^2 e^{-2x}t^3}{3} + 51h^3\xi^3 e^{-3x}t^3 - \frac{176h^3\xi^4 e^{-4x}t^3}{3}. \tag{21}$$

Similarly, $S_m(x,t)$ for $m = 4,5,\cdots$ can be obtained using MATLAB or Mathematical.

Then the series solution expression by q-HAM can be written in the form

$$S(x,t;n;h) \cong S_M(x,t;n;h) = \sum_{j=0}^{M} S_i(x,t;n;h)\left(\frac{1}{n}\right)^i \tag{22}$$

Equation (22) is an appropriate solution to the problem (1) in terms of convergence parameter h and n.

4.3 Numerical Analysis

4.3.1 Comparison Result of VIM and q-HAM

In this subsection, we present graphs of solution to equation (1) obtained by VIM and q-HAM using Mathematica 8 with appropriate $h = -0.6$ and $n = 1$. We use only the 2-term approximate solution

for both VIM and q-HAM.

4.3.2 Comparison of q-HAM, VIM and ADM

We have compared the analytical solution of equation (1) obtained by q-HAM, VIM and that of ADM in [12] presented in Table 1 and Figs. (3) and (4). This is possible when $n=1$ and setting $h=-1.0$. A good correlation is obtained, so expected better convergence if n is larger in the case of q-HAM.

4.3.3 The effect of auxiliary parameter h in the solution obtained by q-HAM compared with that of VIM

In this subsection, we give some numerical analysis on the effect of auxiliary parameter h in the solution obtained by q-HAM compared with the solution obtained by VIM. This effect gives it edge over other analytical methods of solution to non-linear partial differential equations.

Table 1

T	X	S_{q-HAM}	S_{VIM}	S_{ADM}
0.02	0.1000	0.0145	0.0144	0.0146
	0.2000	0.1052	0.1051	0.1052
	0.3000	0.1877	0.1877	0.1878
	0.4000	0.2629	0.2629	0.2629
	0.5000	0.3313	0.3313	0.3313
	0.6000	0.3934	0.3934	0.3934
	0.7000	0.4499	0.4499	0.4499
	0.8000	0.5013	0.5013	0.5012
	0.9000	0.5479	0.5479	0.5479
	1.0000	0.5902	0.5902	0.5902
0.04	0.1000	0.0304	0.0301	0.0315
	0.2000	0.1173	0.1171	0.1178
	0.3000	0.1967	0.1966	0.1969
	0.4000	0.2693	0.2691	0.2692
	0.5000	0.3355	0.3354	0.3354
	0.6000	0.3960	0.3959	0.3958
	0.7000	0.4511	0.4511	0.4510
	0.8000	0.5014	0.5014	0.5013
	0.9000	0.5472	0.5472	0.5471
	1.0000	0.5890	0.5889	0.5888

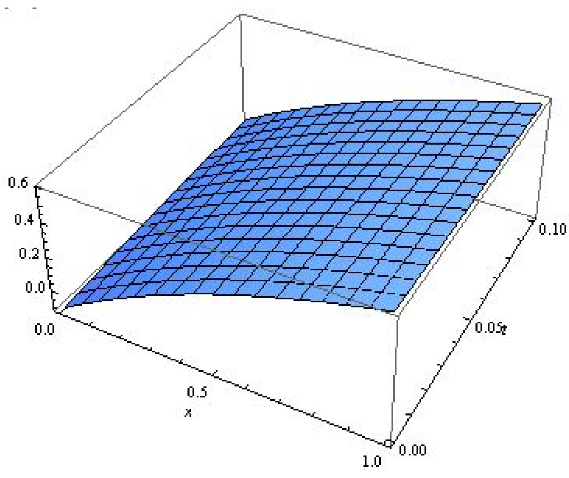

Fig. 1 Solution plot obtained by VIM for $0 \le t \le 0.1$ **and** $0 \le x \le 1$

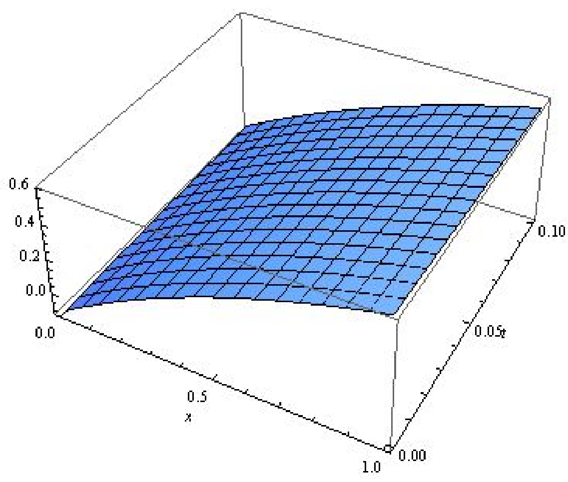

Fig. 2 Solution plot obtained by q-HAM for $0 \le t \le 0.1$ **and** $0 \le x \le 1$

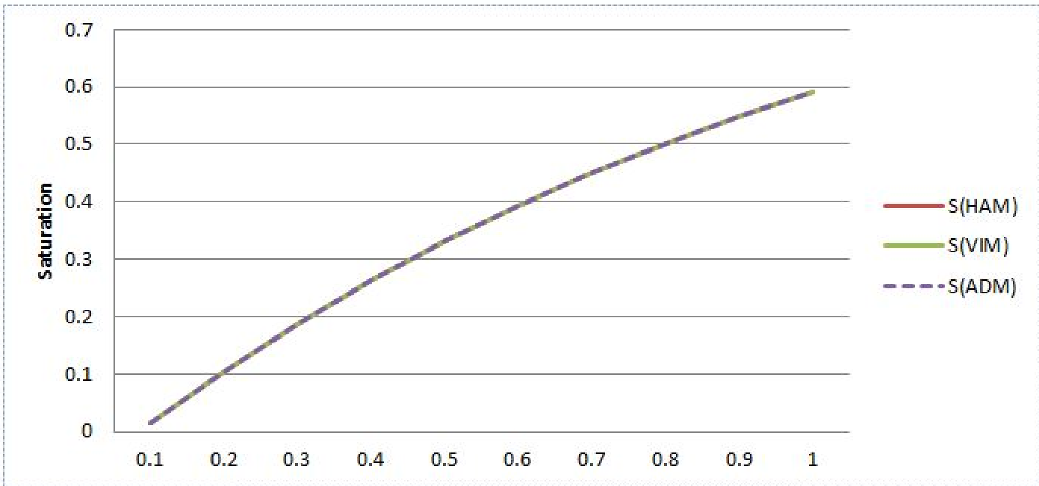

Fig. 3 The Plot of q-HAM, VIM and ADM for fixed t = 0.02 and $0 \le x \le 1$

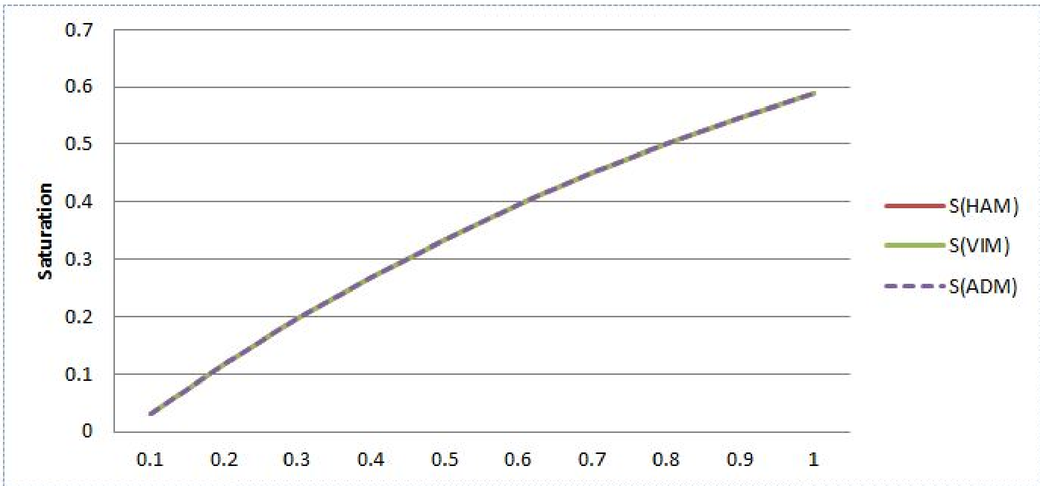

Fig. 4 The Plot of q-HAM, VIM and ADM for fixed t = 0.04 and $0 \le x \le 1$

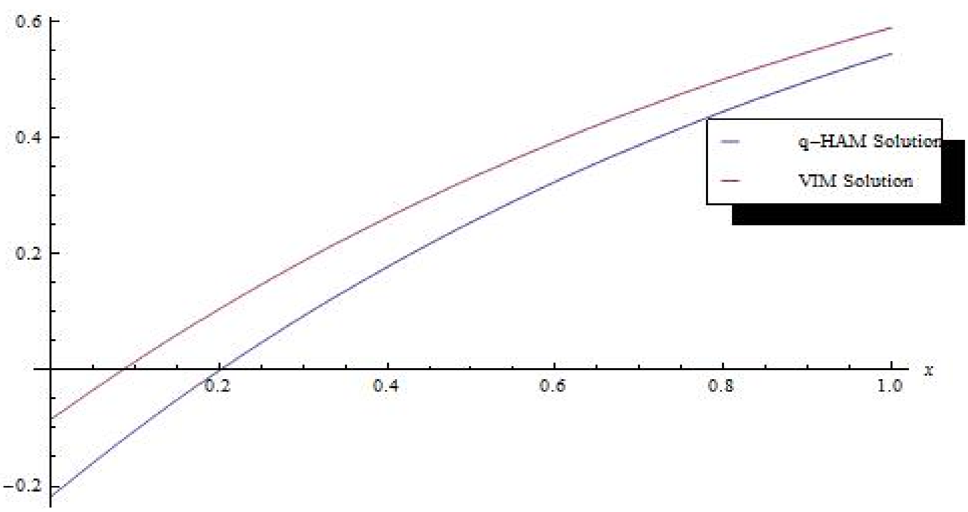

Fig. 5 q-HAM solution and VIM solution for fixed $t = 0.02$, $h = -1.5$ **and** $0 \le x \le 1$

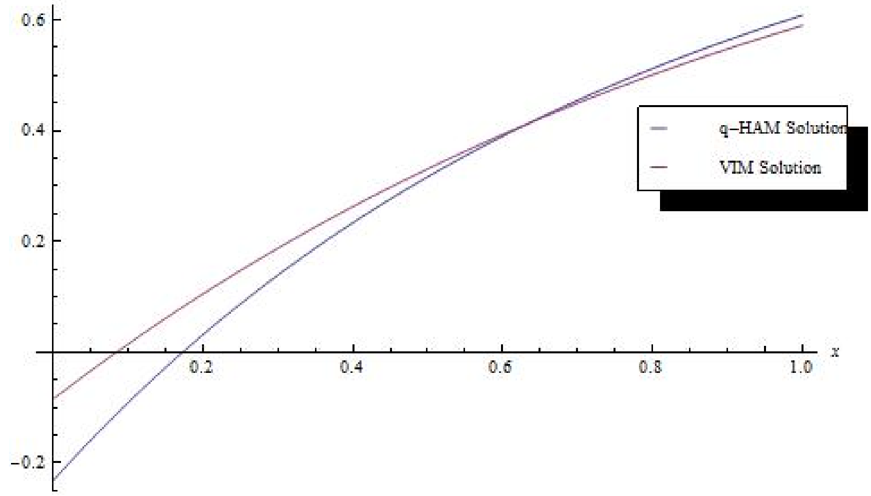

Fig. 6 q-HAM solution and VIM solution for fixed $t = 0.02,\ h = 1.5$ **and** $0 \le x \le 1$

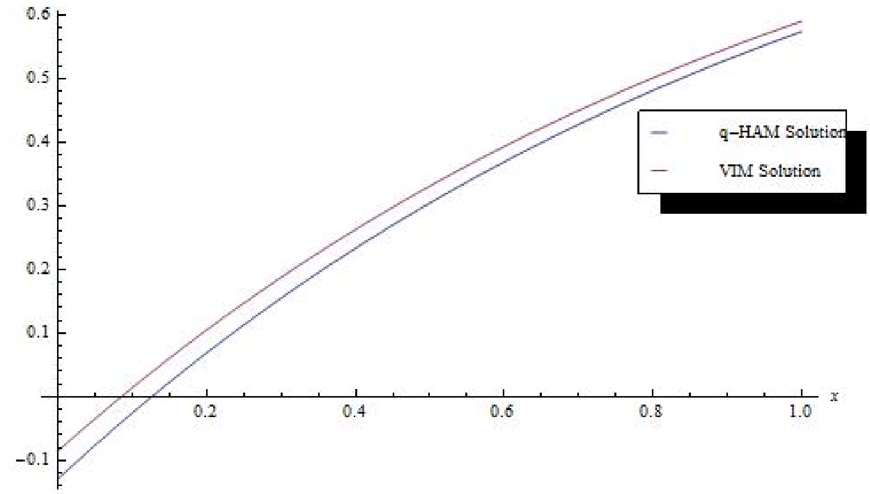

Fig. 7 q-HAM solution and VIM solution for fixed $t = 0.02,\ h = -1.0$ **and** $0 \le x \le 1$

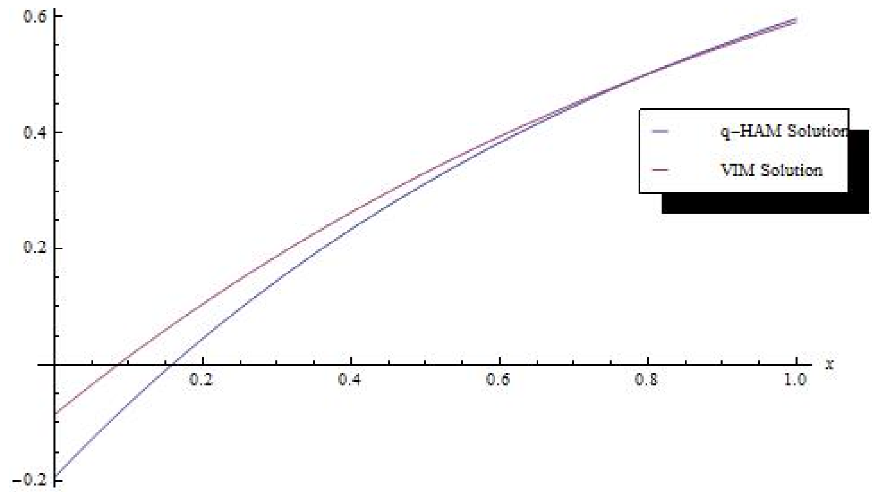

Fig. 8 q-HAM solution and VIM solution for fixed $t = 0.02,\ h = 1.0$ **and** $0 \le x \le 1$

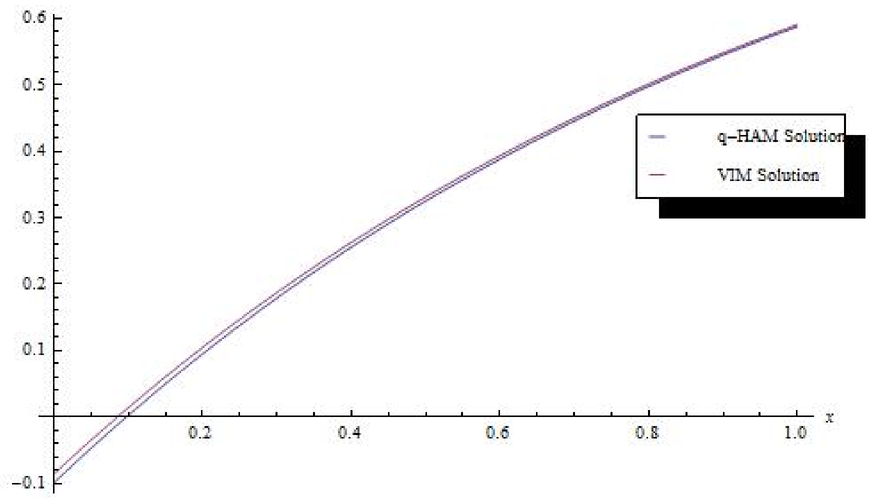

Fig. 9 q-HAM solution and VIM solution for fixed $t=0.02,\ h=-0.5$ **and** $0\le x\le 1$

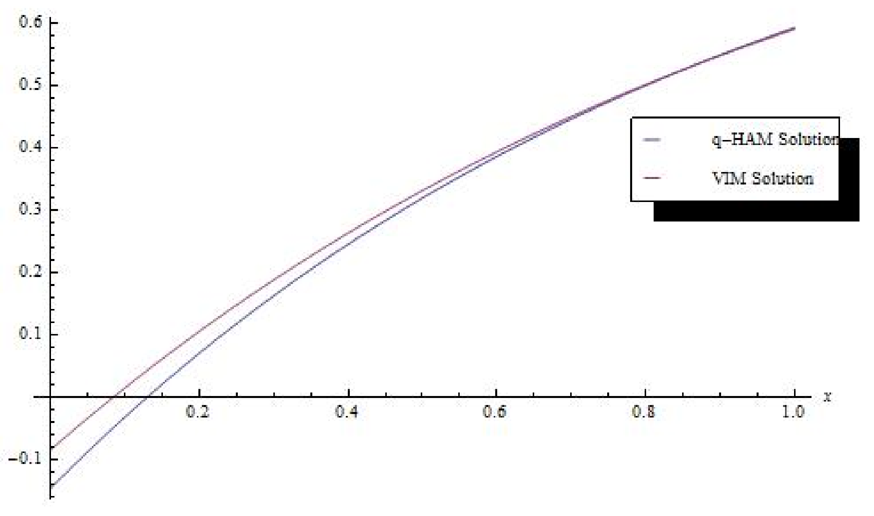

Fig. 10 q-HAM solution and VIM solution for fixed $t=0.02,\ h=0.5$ **and** $0\le x\le 1$

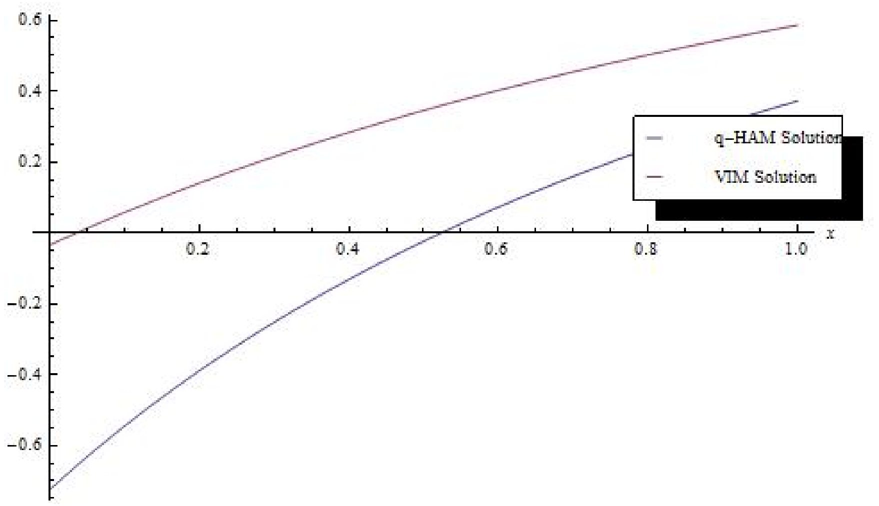

Fig. 11 q-HAM solution and VIM solution for fixed $t=0.1,\ h=-1.5$ **and** $0\le x\le 1$

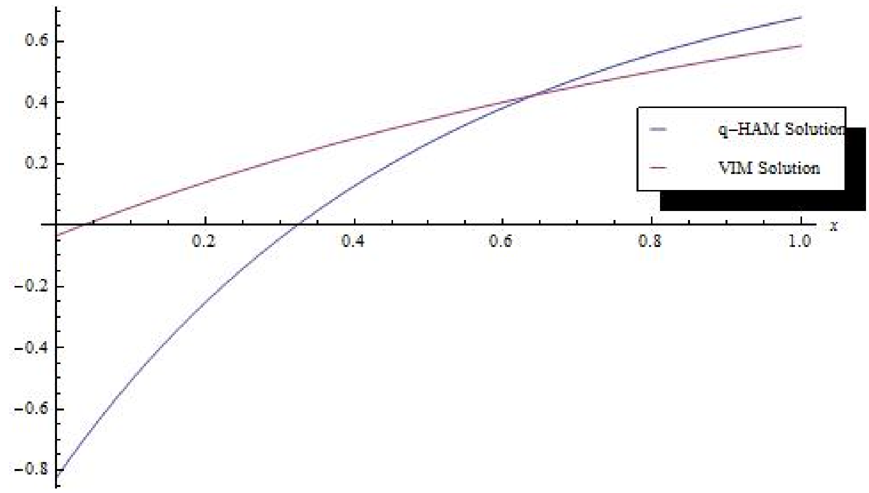

Fig. 12 q-HAM solution and VIM solution for fixed $t = 0.1$, $h = 1.5$ **and** $0 \le x \le 1$

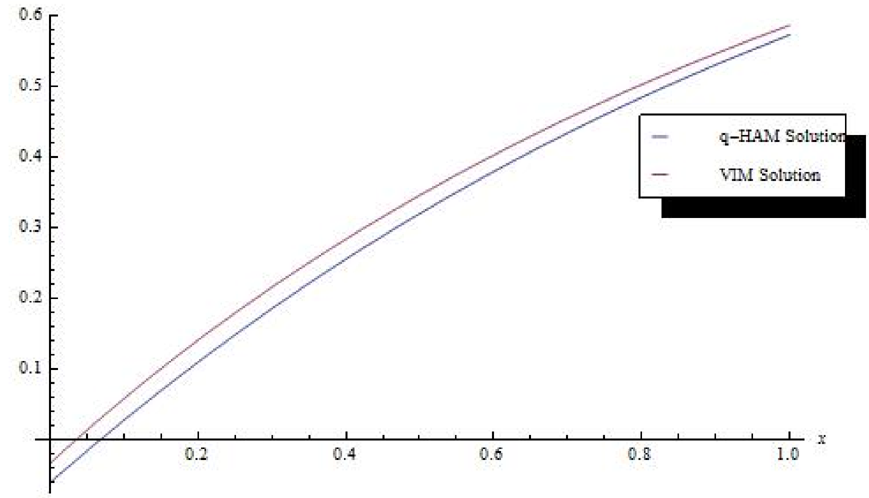

Fig. 13 q-HAM solution and VIM solution for fixed $t = 0.1$, $h = -0.5$ **and** $0 \le x \le 1$

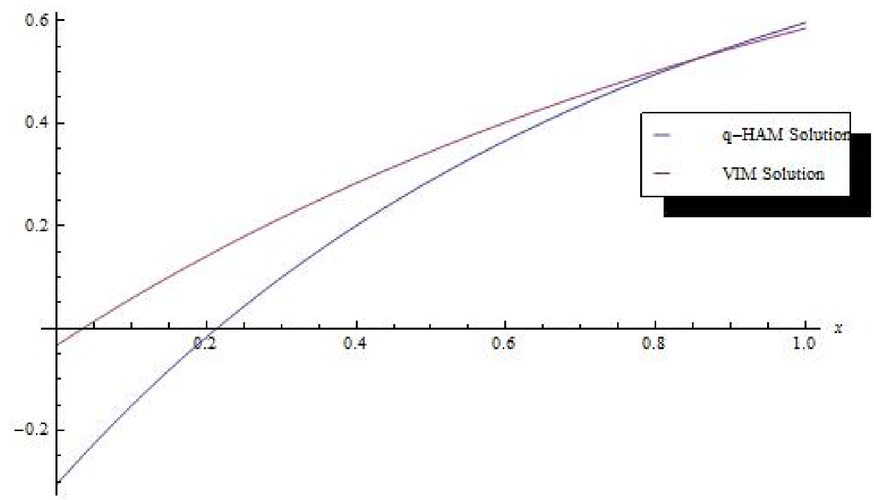

Fig. 14 q-HAM solution and VIM solution for fixed $t = 0.1$, $h = 0.5$ **and** $0 \le x \le 1$.

5. Conclusion

This paper has successfully developed VIM and q-HAM to obtain approximate solution of the nonlinear partial differential equation arising from fingero-imbibition phenomena in double phase flow through porous media. The performance of HAM and other method like ADM is greatly improved by q-HAM shown in figures presented. We have from our results that the presence of fraction factor $\left(\frac{1}{n}\right)^m$ enables the q-HAM to match comparatively with that of VIM. The efficiency and accuracy of these methods is obvious from the graphs, when compared with the result obtained in [6] using ADM.

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Slant Submanifolds of an Almost Hyperbolic Contact Metric Manifolds

Saadet Doğan and Müge Karadağ

Department of Mathematics, Faculty of Arts and Science, Inönü University, 44069 Malatya, Turkey

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Abstract: In the present paper, we introduce the notion of slant submanifolds of an almost hyperbolic contact metric manifolds. We have obtained some results on slant submanifolds of an almost hyperbolic contact metric manifolds. We have given a necessary and sufficient condition for a slant submanifold of an almost hyperbolic contact metric manifolds.

Keywords: Almost hyperbolic contact metric manifolds, slant submanifold

1. Introduction

Almost contact metric manifold with an almost contact metric structure is defined and studied by Blair[1]. On the other hand, almost hyperbolic (ϕ, g, η, ξ) -structure was given and studied by Upadhyay and Dube [7]

The notion of a slant submanifold of an almost Hermitian manifold was given by B.Y.Chen [4]. A. Lotta who has defined and studied slant submanifolds of an almost contact metric manifold [6].Later, L. Cabrerizo and others investigated slant submanifolds of a Sasakian manifolds [3]. The same concept was studied under the name "Slant submanifolds of a Kenmotsu Manifold" by R.S.Gupta, S.M.K.Haider and Mohd.H.Shahid [5].. In 2013, M.Ahmad and K.Ali studied "Semi-invariant submanifold of nearly hyperbolic cosymplectic manifold" and "CR-submanifold of a nearly hyperbolic cosymplectic manifold"[2].

In this paper we study slant submanifolds of an almost hyperbolic contact metric manifolds manifold and we find some interesting results.

Corresponding author: Saadet Doğan, Department of Mathematics, Faculty of Arts and Science, Inönü University. E-mail: saadetdoganmat@gmail.com.

2. Preliminaries

Let $\overline{M}$ be a n-dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric (ϕ, ξ, η, g)-structure, where a tensor ϕ of type $(1,1)$; a vector field ξ and η the dual 1-form of ξ satisfying the followings:

$$\phi^2 X = X + \eta(X)\xi \tag{1}$$

$$g(X, \xi) = \eta(X), \quad \eta(\xi) = -1 \tag{2}$$

$$\phi\xi = 0, \quad \eta \circ \phi = 0 \tag{3}$$

$$g(\phi X, \phi Y) = -g(X, Y) - \eta(X)\eta(Y) \tag{4}$$

$$g(\phi X, Y) = -g(X, \phi Y) \tag{5}$$

for any vector field X and Y in $\overline{TM}$.

Let M be a submanifold immersed in $\overline{M}$ We assume that the vector field ξ is tangent to M For any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$, we write

$$\begin{aligned} \phi X &= PX + FX \\ \phi N &= tN + fN \end{aligned} \tag{6}$$

where $PX(resp.FX)$ denotes the tangential (resp. normal) component of ϕX, and $tN(resp.\, fN)$ denotes the tangential (resp.normal) component of ϕN. If we denote by D the orthogonal distribution to ξ in TM, we can consider the orthogonal direct

decomposition $TM = D \oplus \langle \xi \rangle$.

For each non-zero X tangent to M at x such that X isn't proportional to ξ_x, we denote by $\theta(X)$ the Wirtinger angle of X that is, the angle between ϕX and $T_x M$. Then, M is called slant if $\theta(X)$ is a constant, which is independent of the choice of $x \in M$ and $X \in T_x M - \{\xi_x\}$ [4].

Let ∇ be the Riemannian connection on M Then

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{7}$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \tag{8}$$

for $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$. h and A_N are the second fundamental form and weingarten operator,respectively. h and A_N are related by

$$g(A_N X, Y) = g(h(X,Y), N) \tag{9}$$

for $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$.

3. Slant Submanifolds of Almost Hyperbolic Contact Metric Manifolds

In the present section we prove a characterization theorem for slant submanifolds of an almost hyperbolic contact metric manifolds.

Theorem 1. Let M be a submanifold of an almost hyperbolic contact metric manifold $\overline{M}$ such that $\xi \in \Gamma(TM)$. Then, M is slant if and only if there exist a constant $\lambda \in [0,1]$ such that

$$P^2 = \lambda(I + \eta \otimes \xi) \tag{10}$$

Furthermore, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

Proof: :

$\Rightarrow$): Let M be a slant submanifold of an almost hyperbolic contact metric manifold $\overline{M}$ with slant angle θ. $\xi \in \Gamma(TM)$. Then, from (6)

$$\cos\theta = \frac{g(PX, \phi X)}{\|PX\| \|\phi X\|} \tag{11}$$

$$= \frac{g(PX, PX)}{\|PX\| \|\phi X\|}$$

$$= \frac{\|PX\|}{\|\phi X\|} = const.$$

for $X \in \Gamma(TM)$. In this case, from (11), we have,

$$\|PX\| = \|\phi X\| . \cos\theta \tag{12}$$

If we take the square (12), we get

$$g(PX, PX) = g(\phi X, \phi X) . \cos^2\theta \tag{13}$$

Moreover,

$$\cos\theta = \frac{g(PX, \phi X)}{\|PX\| \|\phi X\|} \tag{14}$$

$$= -\frac{g(X, \phi PX)}{\|PX\| \|\phi X\|}$$

$$= -\frac{g(X, FPX)}{\|PX\| \|\phi X\|} - \frac{g(X, PPX)}{\|PX\| \|\phi X\|}$$

$$= -\frac{g(X, P^2 X)}{\|PX\| \|\phi X\|}$$

all vector field X. Then from (2),(4),(12) and (14) we find

$$\cos\theta . \|PX\| \|\phi X\| = -g(X, P^2 X)$$

$$\cos\theta . \|\phi X\| \|\phi X\| . \cos\theta = -g(X, P^2 X)$$

$$\cos^2\theta . g(\phi X, \phi X) = -g(X, P^2 X)$$

$$\cos^2\theta \{-g(X,X) - \eta(X)\eta(X)\} = -g(P^2 X, X)$$

$$\cos^2\theta \{-g(X + \eta(X)\xi, X)\} = -g(P^2 X, X)$$

In this case, we have

$$P^2 X = \cos^2\theta \{X + \eta(X)\xi\}$$

for any vector field $X \in \Gamma(TM)$. That is, for all $X \in \Gamma(TM)$,

$$P^2 = \lambda(I + \eta \otimes \xi) \tag{15}$$

$$\lambda = \cos^2\theta$$

$\Rightarrow$): Let $P^2 = \lambda(I + \eta \otimes \xi)$. In this case, from

(4) and (10), we find

$$\cos\theta = \frac{g(PX,\phi X)}{\|PX\|\|\phi X\|}$$
$$= -\frac{g(P^2X,X)}{\|PX\|\|\phi X\|}$$
$$= \frac{-\lambda g(X+\eta(X)\xi,X)}{\|PX\|\|\phi X\|}$$
$$= -\lambda\frac{g(X,X)+\eta(X)\eta(X)}{\|PX\|\|\phi X\|}$$
$$= \lambda\frac{g(\phi X,\phi X)}{\|PX\|\|\phi X\|}$$
$$= \lambda\frac{\|\phi X\|}{\|PX\|}$$

for any vector field $X\in\Gamma(TM)$. On the other hand,

$$\cos\theta = \frac{\|PX\|}{\|\phi X\|}$$
$$= \lambda\frac{1}{\cos\theta}$$

then we have

$$\cos^2\theta = \lambda \tag{16}$$

Since λ is constant, θ is constant too. In this case, M is slant.

Corollary 2. Let M be a slant submanifold of an almost hyperbolic contact metric manifold $\overline{M}$, with slant angle θ and $\xi\in\Gamma(TM)$. Then,

$$g(PX,PY) = -\cos^2\theta\{g(X,Y)+\eta(X)\eta(Y)\} \tag{17}$$

$$g(FX,FY) = -\sin^2\theta\{g(X,Y)+\eta(X)\eta(Y)\} \tag{18}$$

for any vector field X and Y in TM.

Proof: We can find with basic calculations using Theorem 3.1.

Theorem 3. Let M be a submanifold of an almost hyperbolic contact metric manifolds $\overline{M}$ such that $\xi\in\Gamma(TM)$. In this case,

$$\overline{\nabla}Q = 0 \tag{19}$$

Proof: From (10), we get

$$QY = \cos^2\theta\{Y+\eta(Y)\xi\} \tag{20}$$

for any vector field Y in TM. If we take the covariant derivative of (17) along X, we get

$$\overline{\nabla}_X(QY) = \cos^2\theta\{\overline{\nabla}_X(Y+\eta(Y)\xi)\} \tag{21}$$
$$= \cos^2\theta\{\overline{\nabla}_XY+\overline{\nabla}_X(\eta(Y)\xi+\eta(Y)\overline{\nabla}_X\xi)\}$$
$$= \cos^2\theta\{\overline{\nabla}_XY+\eta(\overline{\nabla}_XY)\xi\}$$

for $X,Y\in\Gamma(TM)$. On the other hand,

$$(\overline{\nabla}_XQ)Y = \overline{\nabla}_X(QY)-Q\overline{\nabla}_XY$$
$$= \cos^2\theta\{\overline{\nabla}_XY+\eta(\overline{\nabla}_XY)\xi\}-$$
$$-\cos^2\theta\{\overline{\nabla}_XY+\eta(\overline{\nabla}_XY)\xi\} = 0$$

for any vector fields X,Y in $\Gamma(TM)$.

Theorem 4. Let M be a submanifold of an almost hyperbolic contact metric manifolds $\overline{M}$ such that $\xi\in\Gamma(TM)$.Then, the endomorphism $Q|_D$ has only one eigenvalue λ, at each point of M Moreover $\lambda=\cos^2\theta$.

Proof: From (10),

$$QX = P^2X = \cos^2\theta\{X+\eta(X)\xi\} \tag{22}$$
$$= (\cos^2\theta)X$$
$$= \lambda X$$

for all vector field X in TM. From (22), we see that the endomorphism $Q|_D$ has only one eigenvalue.

Theorem 5. Let M be a submanifold of an almost hyperbolic contact metric manifolds $\overline{M}$ such that $\xi\in\Gamma(TM)$.M is slant if and only if

a-) The endomorphism $Q|_D$ has only one eigenvalue at each point of M.

b-) $(\overline{\nabla}_XQ)Y=0$

for any vector fields X and Y.

Proof: $\Rightarrow)$: Suppose that M is slant. Then, statements (a) and (b) follow directly from Theorem 3.2 and Theorem 3.3.

$\Leftarrow)$: Let $\lambda_1(x)$ be the eigenvalue of $Q|_D$ at each

point x of M and $Y \in D$ be a unit eigenvector associated with λ_1, i.e.,

$$QY = \lambda_1 Y \tag{23}$$

If we take the covariant derivative of (23) along X, we find

$$\overline{\nabla}_X (QY) = \overline{\nabla}_X (\lambda_1) Y + \lambda_1 \overline{\nabla}_X Y \tag{24}$$

for any vector field X, Y in TM. From (b), we get

$$(\overline{\nabla}_X Q) Y = \overline{\nabla}_X (QY) - Q\overline{\nabla}_X Y = 0$$

$$\overline{\nabla}_X (QY) = Q\overline{\nabla}_X Y \tag{25}$$

all $X, Y \in \Gamma(TM)$. From (24) and (25), we find

$$Q\overline{\nabla}_X Y = \overline{\nabla}_X (\lambda_1) Y + \lambda_1 \overline{\nabla}_X Y \tag{26}$$

all $X, Y \in \Gamma(TM)$. Since both $\overline{\nabla}_X Y$ and $Q\overline{\nabla}_X Y$ are perpendicular to Y, from (26), we get

$$X(\lambda_1) = 0 \tag{27}$$

for all $X \in \Gamma(TM)$. Then λ_1 is constant on M.

Now we prove that M is slant. In view of (13), it is enough to show that there exists a constant μ such that

$$Q = \mu (I + \eta \otimes \xi)$$

Let X be in TM. Then

$$\overline{X} = X - \eta(X)\xi \in D$$

Hence

$$QX = Q\overline{X}$$

Since

$$Q|_D = \lambda_1 I$$

we have

$$Q\overline{X} = \lambda_1 \overline{X}$$

and so

$$QX = \lambda_1 \overline{X} = \lambda_1 \left(X + \eta(X)\xi \right)$$

Theorem 6. By taking $\mu = \lambda_1$, we get that M is slant. Moreover, if M is slant, $\lambda = \lambda_1 = \mu = \cos^2\theta$, where θ denotes the slant angle of M.

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